NO BULLSHIT

guide to

MATH & PHYSICS

by Ivan Savov

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Contents

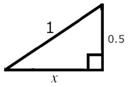
Placement exam

The answers[1] to this placement exam will tell you where to start reading.

1. What is the derivative of $\sin(x)$?

2. What is the second derivative of $A\sin(\omega x)$?

3. What is the value of x ?



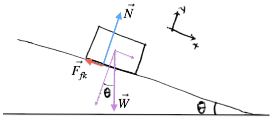
4. What is the magnitude of the gravitational force between two planets of mass M and mass m separated by a distance r?

5. Calculate $\displaystyle\lim_{x\to 3^-} \frac{1}{x-3}$.

6. Solve for t in:

$$7(3+4t) = 11(6t-4).$$

7. What is the component of the weight $\vec{W}$ acting in the x-direction?



8. A mass-spring system is undergoing simple harmonic motion. Its position function is $x(t) = A\sin(\omega t)$. What is its maximum acceleration?

[1] Ans: 1. $\cos(x)$, 2. $-A\omega^2\sin(\omega x)$, 3. $\frac{\sqrt{3}}{2}$, 4. $|\vec{F}_g| = \frac{GMm}{r^2}$, 5. $-\infty$, 6. $\frac{65}{38}$, 7. $+mg\sin\theta$, 8. $A\omega^2$. Key: If you didn't get Q3, Q6 right, you should read the book starting from Chapter 1. If you are mystified by Q1, Q2, Q5, read Chapter 5. If you want to learn how to solve Q4, Q7 and Q8, read Chapter 4.

Concept map

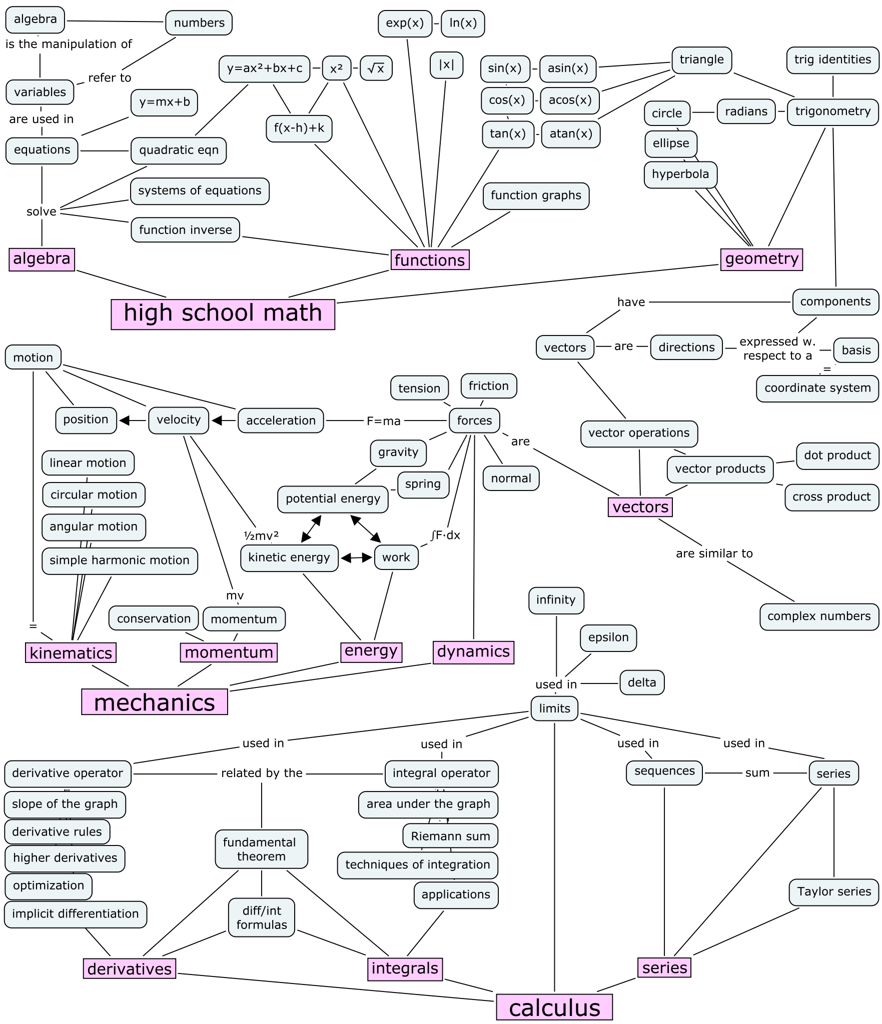


Figure 1: Each concept in this diagram corresponds to one section in the book.

Preface

This book contains lessons on topics in math and physics, written in a style that is jargon-free and to the point. Each lesson covers one concept at the depth required for a first-year university-level course. The main focus of this book is to highlight the intricate connections between the concepts of math and physics. Seeing the similarities and parallels between the concepts is the key to understanding.

Why?

The genesis of this book dates back to my student days when I was required to purchase expensive textbooks for my courses. Not only are these textbooks expensive, they are also tedious to read. Who has the energy to go through thousands of pages of explanations? I began to wonder, "What's the deal with these thick books?" Later, I realized mainstream textbooks are long because the textbook industry wants to make more profits. You don't need to read 1000 pages to learn calculus; the numerous full-page colour pictures and the repetitive text that are used to "pad" calculus textbooks are there to make the $130 price seem reasonable.

Looking at this situation, I said to myself, "something must be done," and I sat down and wrote a modern textbook to explain math and physics clearly, concisely, and affordably. There was no way I was going to let mainstream publishers ruin the learning experience of these beautiful subjects for the next generation of students.

How?

The sections in this book are **self-contained tutorials**. Each section covers the definitions, formulas, and explanations associated with a single topic. You can therefore read the sections in any order you find logical. Along the way, you will learn about the *connections* between the concepts of calculus and mechanics. Understanding mechanics is much easier if you know the ideas of calculus. At the same time, the ideas behind calculus are best illustrated through concrete physics

examples. Learning the two subjects simultaneously is the best approach.

In order to make the study of mechanics and calculus accessible for all readers, we'll begin with a review chapter on numbers, algebra, equations, functions, and other prerequisite concepts. If you feel a little rusty on those concepts, be sure to check out Chapter 1.

Each chapter ends with a section of **practice problems** designed to test your understanding of the concepts developed in that chapter. Make sure you spend plenty of time on these problems to practice what you've learned. Figuring out how to use an equation on your own in the process of solving a problem is a much more valuable experience than simply memorizing the equation.

For optimal learning efficiency, I recommend that you spend as much time working through the practice problems as you will spend reading the lessons. The problems you find difficult to solve will tell you which sections of the chapter you need to revisit. An additional benefit of testing your skills on the practice problems is that you'll be prepared in case a teacher ever tries to test you.

Throughout the book, I've included **links to Internet resources** like animations, demonstrations, and webpages with further reading material. Once you understand the basics, you'll be able to understand far more Internet resources. The links provided are a starting point for further exploration.

Is this book for you?

My aim is to make learning calculus and mechanics more accessible. Anyone should be able to open this book and become proficient in calculus and mechanics, regardless of their mathematical background.

The book's primary intended audience is students. Students taking a mechanics class can read the chapters sequentially until Chapter 4, and optionally read Chapter 5 for general knowledge. Taking a calculus course? Skip ahead directly to the calculus chapter (Chapter 5). High school students or university students taking a precalculus class will benefit from reading Chapter 1, which is a concise but thorough review of fundamental math concepts like numbers, equations, functions, and trigonometry.

MECH CLASS	CALC CLASS	PRECALC CLASS
Ch. 1	Ch. 1	Ch. 1
Ch. 2	Ch. 2	Ch. 2†
Ch. 3		
Ch. 4		
Ch. 5†	Ch. 5	

† = optional reading.

Non-students, don't worry: you don't need to be taking a class in order to learn math. Independent learners interested in learning

university-level material will find this book very useful. Many university graduates read this book to remember the calculus they learned back in their university days.

In general, anyone interested in rekindling their relationship with mathematics should consider this book as an opportunity to repair the broken connection. Math is good stuff; you shouldn't miss out on it. People who think they absolutely *hate* math should read Chapter 1 as therapy.

About the author

I have been teaching math and physics for more than 10 years as a private tutor. My tutoring experience has taught me how to explain concepts that people find difficult to understand. I've had the chance to experiment with different approaches for explaining challenging material. Fundamentally, I've learned from teaching that understanding connections between concepts is much more important than memorizing facts. It's not about how many equations you know, but about knowing how to get from one equation to another.

I completed my undergraduate studies at McGill University in electrical engineering, then did a M.Sc. in physics, and recently completed a Ph.D. in computer science. In my career as a researcher, I've been fortunate to learn from very inspirational teachers, who had the ability to distill the essential ideas and explain things in simple language. With my writing, I want to recreate the same learning experience for you. I founded the `Minireference Co.` to revolutionize the textbook industry. We make textbooks that don't suck.

Ivan Savov
Montreal, 2014

Introduction

The last two centuries have been marked by tremendous technological advances. Every sector of the economy has been transformed by the use of computers and the advent of the Internet. There is no doubt technology's importance will continue to grow in the coming years.

The best part is that you don't need to know how technology works to use it. You need not understand how Internet protocols operate to check your email and find original pirate material. You don't need to be a programmer to tell a computer to automate repetitive tasks and increase your productivity. However, when it comes to building *new* things, understanding becomes important. One particularly useful skill is the ability to create mathematical models of real-world situations. The techniques of mechanics and calculus are powerful building blocks for understanding the world around us. This is why these courses are taught in the first year of university studies: they contain keys that unlock the rest of science.

Calculus and mechanics can be difficult subjects. Understanding the material isn't hard *per se*, but it takes patience and practice. Calculus and mechanics become much easier to absorb when you break down the material into manageable chunks. It is most important you learn the *connections* between concepts.

Before we start with the equations, it's worthwhile to preview the material covered in this book. After all, you should know what kind of trouble you're getting yourself into.

Chapter 1 is a comprehensive review of math fundamentals including algebra, equation solving, and functions. The exposition of each topic is brief to make for easy reading. This chapter is highly recommended for readers who haven't looked at math recently; if you need a refresher on math, Chapter 1 is for you. It is extremely important to firmly grasp the basics. What is $\sin(0)$? What is $\sin(\pi/4)$? What does the graph of $\sin(x)$ look like? Arts students interested in enriching their cultural insight with knowledge that is 2000+ years old can read this chapter as therapy to recover from any damaging educational experiences they may have encountered in high school.

In Chapter 2, we'll look at how techniques of high school math can be used to describe and model the world. We'll learn about the basic laws that govern the motion of objects in one dimension and the mathematical equations that describe the motion. By the end of this chapter, you'll be able to predict the flight time of a ball thrown in the air.

In Chapter 3, we will learn about vectors. Vectors describe directional quantities like forces and velocities. We need vectors to properly understand the laws of physics. Vectors are used in many areas of science and technology, so becoming comfortable with vector calculations will pay dividends when learning other subjects.

Chapter 4 is all about mechanics. We'll study the motion of objects, predict their future trajectories, and learn how to use abstract concepts like momentum and energy. Science students who "hate" physics can study this chapter to learn how to use the 20 main equations and laws of physics. You'll see physics is actually quite simple.

Chapter 5 covers topics from differential calculus and integral calculus. We will study limits, derivatives, integrals, sequences, and series. You'll find that 100 pages are enough to cover all the concepts in calculus, as well as illustrate them with examples and practice exercises.

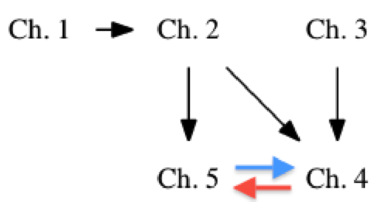


Figure 2: The prerequisite structure for the chapters in this book.

Calculus and mechanics are often taught as separate subjects. It shouldn't be like that! If you learn calculus without mechanics, it will be boring. If you learn physics without calculus, you won't truly understand. The exposition in this book covers both subjects in an integrated manner and aims to highlight the connections between them. Let's dig in.

Chapter 1

Math fundamentals

In this chapter we'll review the fundamental ideas of mathematics, including numbers, equations, and functions. To understand college-level textbooks, you need to be comfortable with mathematical calculations. Many people have trouble with math, however. Some people say they *hate* math, or could never learn it. It's not uncommon for children who score poorly on their school math exams to develop math complexes in their grown lives. If you are carrying any such emotional baggage, you can drop it right here and right now.

Do NOT worry about math! You are an adult, and you can learn math much more easily than when you were in high school. We'll review *everything* you need to know about high school math, and by the end of this chapter, you'll see that math is nothing to worry about.

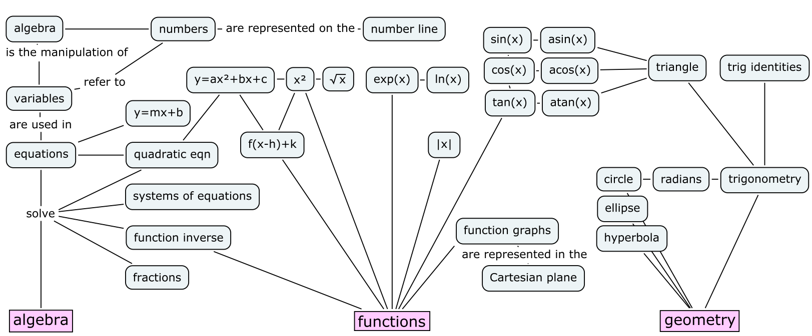


Figure 1.1: A concept map showing the mathematical topics that we will cover in this chapter. We'll learn how to solve equations using algebra, how to model the world using functions, and how to think geometrically. The material in this chapter is required for your understanding of the more advanced topics in this book.

1.1 Solving equations

Most math skills boil down to being able to manipulate and solve equations. Solving an equation means finding the value of the unknown in the equation.

Check this shit out:

$$x^2 - 4 = 45.$$

To solve the above equation is to answer the question "What is x?" More precisely, we want to find the number that can take the place of x in the equation so that the equality holds. In other words, we're asking,

"Which number times itself minus four gives 45?"

That is quite a mouthful, don't you think? To remedy this verbosity, mathematicians often use specialized mathematical symbols. The problem is that these specialized symbols can be very confusing. Sometimes even the simplest math concepts are inaccessible if you don't know what the symbols mean.

What are your feelings about math, dear reader? Are you afraid of it? Do you have anxiety attacks because you think it will be too difficult for you? Chill! Relax, my brothers and sisters. There's nothing to it. Nobody can magically guess what the solution to an equation is immediately. To find the solution, you must break the problem down into simpler steps.

To find x, we can manipulate the original equation, transforming it into a different equation (as true as the first) that looks like this:

$$x = \text{only numbers}.$$

That's what it means to *solve*. The equation is solved because you can type the numbers on the right-hand side of the equation into a calculator and obtain the numerical value of x that you're seeking.

By the way, before we continue our discussion, let it be noted: the equality symbol (=) means that all that is to the left of = is equal to all that is to the right of =. To keep this equality statement true, **for every change you apply to the left side of the equation, you must apply the same change to the right side of the equation**.

To find x, we need to correctly manipulate the original equation into its final form, simplifying it in each step. The only requirement is that the manipulations we make transform one true equation into another true equation. Looking at our earlier example, the first simplifying step is to add the number four to both sides of the equation:

$$x^2 - 4 + 4 = 45 + 4,$$

which simplifies to

$$x^2 = 49.$$

The expression looks simpler, yes? How did I know to perform this operation? I was trying to "undo" the effects of the operation -4. We undo an operation by applying its *inverse*. In the case where the operation is subtraction of some amount, the inverse operation is the addition of the same amount. We'll learn more about function inverses in Section 1.4 (page 12).

We're getting closer to our goal, namely to *isolate* x on one side of the equation, leaving only numbers on the other side. The next step is to undo the square x^2 operation. The inverse operation of squaring a number x^2 is to take the square root $\sqrt{}$ so this is what we'll do next. We obtain

$$\sqrt{x^2} = \sqrt{49}.$$

Notice how we applied the square root to both sides of the equation? If we don't apply the same operation to both sides, we'll break the equality!

The equation $\sqrt{x^2} = \sqrt{49}$ simplifies to

$$|x| = 7.$$

What's up with the vertical bars around x? The notation $|x|$ stands for the *absolute value* of x, which is the same as x except we ignore the sign. For example $|5| = 5$ and $|-5| = 5$, too. The equation $|x| = 7$ indicates that both $x = 7$ and $x = -7$ satisfy the equation $x^2 = 49$. Seven squared is 49, and so is $(-7)^2 = 49$ because two negatives cancel each other out.

We're done since we isolated x. The final solutions are

$$x = 7 \qquad \text{or} \qquad x = -7.$$

Yes, there are *two* possible answers. You can check that both of the above values of x satisfy the initial equation $x^2 - 4 = 45$.

If you are comfortable with all the notions of high school math and you feel you could have solved the equation $x^2 - 4 = 45$ on your own, then you should consider skipping ahead to Chapter 2. If on the other hand you are wondering how the squiggle killed the power two, then this chapter is for you! In the following sections we will review all the essential concepts from high school math that you will need to power through the rest of this book. First, let me tell you about the different kinds of numbers.

Compact notation

Symbolic manipulation is a powerful tool because it allows us to manage complexity. Say you're solving a physics problem in which you're told the mass of an object is $m = 140$ kg. If there are many steps in the calculation, would you rather use the number 140 kg in each step, or the shorter variable m? It's much easier in the long run to use the variable m throughout your calculation, and wait until the last step to substitute the value 140 kg when computing the final answer.

1.4 Functions and their inverses

As we saw in the section on solving equations, the ability to "undo" functions is a key skill for solving equations.

Example Suppose we're solving for x in the equation

$$f(x) = c,$$

where f is some function and c is some constant. Our goal is to isolate x on one side of the equation, but the function f stands in our way.

By using the inverse function (denoted f^{-1}) we "undo" the effects of f. Then we apply the inverse function f^{-1} to both sides of the equation to obtain

$$f^{-1}(f(x)) = x = f^{-1}(c).$$

By definition, the inverse function f^{-1} performs the opposite action of the function f so together the two functions cancel each other out. We have $f^{-1}(f(x)) = x$ for any number x.

Provided everything is kosher (the function f^{-1} must be defined for the input c), the manipulation we made above is valid and we have obtained the answer $x = f^{-1}(c)$.

The above example introduces the notation f^{-1} for denoting the function's *inverse*. This notation is borrowed from the notion of inverse numbers: multiplication by the number a^{-1} is the inverse operation of multiplication by the number a: $a^{-1}ax = 1x = x$. In the case of functions, however, the negative-one exponent does not refer to "one over-$f(x)$" as in $\frac{1}{f(x)} = (f(x))^{-1}$; rather, it refers to the function's inverse. In other words, the number $f^{-1}(y)$ is equal to the number x such that $f(x) = y$.

Be careful: sometimes applying the inverse leads to multiple solutions. For example, the function $f(x) = x^2$ maps two input values (x and $-x$) to the same output value $x^2 = f(x) = f(-x)$. The inverse function of $f(x) = x^2$ is $f^{-1}(x) = \sqrt{x}$, but both $x = +\sqrt{c}$

and $x = -\sqrt{c}$ are solutions to the equation $x^2 = c$. In this case, this equation's solutions can be indicated in shorthand notation as $x = \pm\sqrt{c}$.

Formulas

Here is a list of common functions and their inverses:

$$
\begin{aligned}
\text{function } f(x) \quad &\Leftrightarrow \quad \text{inverse } f^{-1}(x) \\
x+2 \quad &\Leftrightarrow \quad x-2 \\
2x \quad &\Leftrightarrow \quad \frac{1}{2}x \\
-x \quad &\Leftrightarrow \quad -x \\
x^2 \quad &\Leftrightarrow \quad \pm\sqrt{x} \\
2^x \quad &\Leftrightarrow \quad \log_2(x) \\
3x+5 \quad &\Leftrightarrow \quad \frac{1}{3}(x-5) \\
a^x \quad &\Leftrightarrow \quad \log_a(x) \\
\exp(x) \equiv e^x \quad &\Leftrightarrow \quad \ln(x) \equiv \log_e(x) \\
\sin(x) \quad &\Leftrightarrow \quad \sin^{-1}(x) \equiv \arcsin(x) \\
\cos(x) \quad &\Leftrightarrow \quad \cos^{-1}(x) \equiv \arccos(x)
\end{aligned}
$$

The function-inverse relationship is *reflexive*—if you see a function on one side of the above table (pick a side, any side), you'll find its inverse on the opposite side.

Example

Let's say your teacher doesn't like you and right away, on the first day of class, he gives you a serious equation and tells you to find x:

$$\log_5\left(3 + \sqrt{6\sqrt{x} - 7}\right) = 34 + \sin(5.5) - \Psi(1).$$

See what I mean when I say the teacher doesn't like you?

First, note that it doesn't matter what Ψ (the capital Greek letter *psi*) is, since x is on the other side of the equation. You can keep copying $\Psi(1)$ from line to line, until the end, when you throw the ball back to the teacher. "My answer is in terms of *your* variables, dude. *You* go figure out what the hell Ψ is since you brought it up in the first place!" By the way, it's not actually recommended to quote

me verbatim should a situation like this arise. The same goes with $\sin(5.5)$. If you don't have a calculator handy, don't worry about it. Keep the expression $\sin(5.5)$ instead of trying to find its numerical value. In general, try to work with variables as much as possible and leave the numerical computations for the last step.

Okay, enough beating about the bush. Let's just find x and get it over with! On the right-hand side of the equation, we have the sum of a bunch of terms with no x in them, so we'll leave them as they are. On the left-hand side, the outermost function is a logarithm base 5. Cool. Looking at the table of inverse functions we find the exponential function is the inverse of the logarithm: $a^x \Leftrightarrow \log_a(x)$. To get rid of $\log_5$, we must apply the exponential function base 5 to both sides:

$$5^{\log_5\left(3+\sqrt{6\sqrt{x}-7}\right)} = 5^{34+\sin(5.5)-\Psi(1)},$$

which simplifies to

$$3 + \sqrt{6\sqrt{x} - 7} = 5^{34+\sin(5.5)-\Psi(1)},$$

since 5^x cancels $\log_5 x$.

From here on, it is going to be as if Bruce Lee walked into a place with lots of bad guys. Addition of 3 is undone by subtracting 3 on both sides:

$$\sqrt{6\sqrt{x} - 7} = 5^{34+\sin(5.5)-\Psi(1)} - 3.$$

To undo a square root we take the square:

$$6\sqrt{x} - 7 = \left(5^{34+\sin(5.5)-\Psi(1)} - 3\right)^2.$$

Add 7 to both sides,

$$6\sqrt{x} = \left(5^{34+\sin(5.5)-\Psi(1)} - 3\right)^2 + 7,$$

divide by 6

$$\sqrt{x} = \frac{1}{6}\left(\left(5^{34+\sin(5.5)-\Psi(1)} - 3\right)^2 + 7\right),$$

and square again to find the final answer:

$$x = \left[\frac{1}{6}\left(\left(5^{34+\sin(5.5)-\Psi(1)} - 3\right)^2 + 7\right)\right]^2.$$

Did you see what I was doing in each step? Next time a function stands in your way, hit it with its inverse so it knows not to challenge you ever again.

Discussion

The recipe I have outlined above is not universally applicable. Sometimes x isn't alone on one side. Sometimes x appears in several places in the same equation. In these cases, you can't effortlessly work your way, Bruce Lee-style, clearing bad guys and digging toward x—you need other techniques.

The bad news is there's no general formula for solving complicated equations. The good news is the above technique of "digging toward x" is sufficient for 80% of what you are going to be doing. You can get another 15% if you learn how to solve the quadratic equation (page 23):

$$ax^2 + bx + c = 0.$$

Solving third-degree polynomial equations like $ax^3 + bx^2 + cx + d = 0$ with pen and paper is also possible, but at this point you might as well start using a computer to solve for the unknowns.

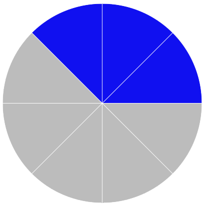
There are all kinds of other equations you can learn how to solve: equations with multiple variables, equations with logarithms, equations with exponentials, and equations with trigonometric functions. The principle of "digging" toward the unknown by applying inverse functions is the key for solving all these types of equations, so be sure to practice using it.

1.5 Fractions

The set of rational numbers $\mathbb{Q}$ is the set of numbers that can be written as a *fraction* of two integers:

$$\mathbb{Q} \equiv \left\{ \frac{m}{n} \;\middle|\; m \text{ and } n \text{ are in } \mathbb{Z} \text{ and } n \neq 0 \right\},$$

where $\mathbb{Z}$ denotes the set of integers $\mathbb{Z} \equiv \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.



Fractions describe what happens when a *whole* is cut into n equal parts and we are given m of those parts. For example, the fraction $\frac{3}{8}$ describes having three parts out of a whole cut into eight parts, hence the name "three eighths." We read $\frac{1}{4}$ either as *one over four* or *one quarter*, which is also equal to 0.25, but as you can see the notation $\frac{1}{4}$ is more compact and nicer. Why nicer? Check out these simple fractions:

$$\frac{1}{1} = 1.0$$
$$\frac{1}{2} = 0.5$$

1.6 Basic rules of algebra

It's important that you know the general rules for manipulating numbers and variables, a process otherwise known as—you guessed it—*algebra.* This little refresher will cover these concepts to make sure you're comfortable on the algebra front. We'll also review some important algebraic tricks, like *factoring* and *completing the square*, which are useful when solving equations.

When an expression contains multiple things added together, we call those things *terms.* Furthermore, terms are usually composed of many things multiplied together. When a number x is obtained as the product of other numbers like $x = abc$, we say "x factors into a, b, and c." We call a, b, and c the *factors* of x.

Given any four numbers a, b, c, and d, we can apply the following algebraic properties:

1. Associative property: $a + b + c = (a + b) + c = a + (b + c)$ and $abc = (ab)c = a(bc)$
2. Commutative property: $a + b = b + a$ and $ab = ba$
3. Distributive property: $a(b + c) = ab + ac$

We use the distributive property every time we *expand* brackets. For example $a(b + c + d) = ab + ac + ad$. The brackets, also known as parentheses, indicate the expression $(b + c + d)$ must be treated as a whole: a factor that consists of three terms. Multiplying this expression by a is the same as multiplying each term by a.

The opposite operation of expanding is called *factoring*, which consists of rewriting the expression with the common parts taken out in front of a bracket: $ab + ac = a(b + c)$. In this section, we'll discuss both of these operations and illustrate what they're capable of.

Expanding brackets

The distributive property is useful when dealing with polynomials:

$$(x + 3)(x + 2) = x(x + 2) + 3(x + 2) = x^2 + x2 + 3x + 6.$$

We can use the commutative property on the second term $x2 = 2x$, then combine the two x terms into a single term to obtain

$$(x + 3)(x + 2) = x^2 + 5x + 6.$$

Let's look at this operation in its abstract form:

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

The product of two linear terms (expressions of the form $x + ?$) is equal to a quadratic expression. Observe that the middle term on the right-hand side contains the *sum* of the two constants on the left-hand side $(a + b)$, while the third term contains their product ab.

It is very common for people to confuse these terms. If you are ever confused about an algebraic expression, go back to the distributive property and expand the expression using a step-by-step approach. As a second example, consider this slightly-more-complicated algebraic expression and its expansion:

$$\begin{aligned}(x + a)(bx^2 + cx + d) &= x(bx^2 + cx + d) + a(bx^2 + cx + d)\\ &= bx^3 + cx^2 + dx + abx^2 + acx + ad\\ &= bx^3 + (c + ab)x^2 + (d + ac)x + ad.\end{aligned}$$

Note how all terms containing x^2 are grouped into a one term, and all terms containing x are grouped into another term. We use this pattern when dealing with expressions containing different powers of x.

Example Suppose we are asked to solve for t in the equation

$$7(3 + 4t) = 11(6t - 4).$$

Since the unknown t appears on both sides of the equation, it is not immediately obvious how to proceed.

To solve for t, we must bring all t terms to one side and all constant terms to the other side. First, expand the two brackets to obtain

$$21 + 28t = 66t - 44.$$

Then move things around to relocate all ts to the equation's right-hand side and all constants to the left-hand side:

$$21 + 44 = 66t - 28t.$$

We see t is contained in both terms on the right-hand side, so we can rewrite the equation as

$$21 + 44 = (66 - 28)t.$$

The answer is within close reach: $t = \frac{21+44}{66-28} = \frac{65}{38}$.

Factoring

Factoring involves taking out the common part(s) of a complicated expression in order to make the expression more compact. Suppose you're given the expression $6x^2y + 15x$ and must simplify it by taking

out common factors. The expression has two terms and each term can be split into its constituent factors to obtain

$$6x^2y + 15x = (3)(2)(x)(x)y + (5)(3)x.$$

Since factors x and 3 appear in both terms, we can *factor them out* to the front like this:

$$6x^2y + 15x = 3x(2xy + 5).$$

The expression on the right shows $3x$ is common to both terms.

Here's another example where factoring is used:

$$2x^2y + 2x + 4x = 2x(xy + 1 + 2) = 2x(xy + 3).$$

Quadratic factoring

When dealing with a quadratic function, it is often useful to rewrite the function as a product of two factors. Suppose you're given the quadratic function $f(x) = x^2 - 5x + 6$ and asked to describe its properties. What are the *roots* of this function? In other words, for what values of x is this function equal to zero? For which values of x is the function positive, and for which x values is the function negative?

Factoring the expression $x^2 - 5x + 6$ will help us see the properties of the function more clearly. To *factor* a quadratic expression is to express it as the product of two factors:

$$f(x) = x^2 - 5x + 6 = (x - 2)(x - 3).$$

We now see at a glance the solutions (roots) are $x_1 = 2$ and $x_2 = 3$. We can also see for which x values the function will be overall positive: for $x > 3$, both factors will be positive, and for $x < 2$ both factors will be negative, and a negative times a negative gives a positive. For values of x such that $2 < x < 3$, the first factor will be positive, and the second factor negative, making the overall function negative.

For certain simple quadratics like the one above, you can simply *guess* what the factors will be. For more complicated quadratic expressions, you'll need to use the quadratic formula (page 23), which will be the subject of the next section. For now let us continue with more algebra tricks.

Completing the square

Any quadratic expression $Ax^2 + Bx + C$ can be rewritten in the form $A(x - h)^2 + k$ for some constants h and k. This process is called *completing the square* due to the reasoning we follow to find the value

1.8 Exponents

In math we must often multiply together the same number many times, so we use the notation

$$b^n = \underbrace{bbb\cdots bb}_{n \text{ times}}$$

to denote some number b multiplied by itself n times. In this section we'll review the basic terminology associated with exponents and discuss their properties.

Definitions

The fundamental ideas of exponents are:

- b^n: the number b raised to the power n
  - ▷ b: the *base*
  - ▷ n: the *exponent* or *power* of b in the expression b^n

By definition, the zeroth power of any number is equal to one, expressed as $b^0 = 1$.

We'll also discuss *exponential functions* of the form $f : \mathbb{R} \to \mathbb{R}$. In particular, we define the following important exponential functions:

- b^x: the exponential function base b
- 10^x: the exponential function base 10
- $\exp(x) \equiv e^x$: the exponential function base e. The number e is called *Euler's number*.
- 2^x: the exponential function base 2. This function is very important in computer science.

The number $e = 2.7182818\ldots$ is a special base with many applications. We call e the *natural* base.

Another special base is 10 because we use the decimal system for our numbers. We can write very large numbers and very small numbers as powers of 10. For example, one thousand can be written as $1\,000 = 10^3$, one million is $1\,000\,000 = 10^6$, and one billion is $1\,000\,000\,000 = 10^9$.

Formulas

The following properties follow from the definition of exponentiation as repeated multiplication.

Property 1 Multiplying together two exponential expressions that have the same base is the same as adding the exponents:

$$b^m b^n = \underbrace{bbb\cdots bb}_{m \text{ times}}\underbrace{bbb\cdots bb}_{n \text{ times}} = \underbrace{bbbbbbb\cdots bb}_{m+n \text{ times}} = b^{m+n}.$$

Property 2 Division by a number can be expressed as an exponent of minus one:

$$b^{-1} \equiv \frac{1}{b}\,.$$

A negative exponent corresponds to a division:

$$b^{-n} = \frac{1}{b^n}\,.$$

Property 3 By combining Property 1 and Property 2 we obtain the following rule:

$$\frac{b^m}{b^n} = b^{m-n}.$$

In particular we have $b^n b^{-n} = b^{n-n} = b^0 = 1$. Multiplication by the number b^{-n} is the inverse operation of multiplication by the number b^n. The net effect of the combination of both operations is the same as multiplying by one, i.e., the identity operation.

Property 4 When an exponential expression is exponentiated, the inner exponent and the outer exponent multiply:

$$(b^m)^n = \underbrace{(\underbrace{bbb\cdots bb}_{m \text{ times}})(\underbrace{bbb\cdots bb}_{m \text{ times}})\cdots(\underbrace{bbb\cdots bb}_{m \text{ times}})}_{n \text{ times}} = b^{mn}.$$

Property 5.1

$$(ab)^n = \underbrace{(ab)(ab)(ab)\cdots(ab)(ab)}_{n \text{ times}} = \underbrace{aaa\cdots aa}_{n \text{ times}}\underbrace{bbb\cdots bb}_{n \text{ times}} = a^n b^n.$$

Property 5.2

$$\left(\frac{a}{b}\right)^n = \underbrace{\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\cdots\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)}_{n \text{ times}} = \frac{\overbrace{aaa\cdots aa}^{n \text{ times}}}{\underbrace{bbb\ \cdots\ bb}_{n \text{ times}}} = \frac{a^n}{b^n}.$$

Property 6 Raising a number to the power $\frac{1}{n}$ is equivalent to finding the n^{th} root of the number:

$$b^{\frac{1}{n}} \equiv \sqrt[n]{b}.$$

In particular, the square root corresponds to the exponent of one half: $\sqrt{b} = b^{\frac{1}{2}}$. The cube root (the inverse of x^3) corresponds to $\sqrt[3]{b} \equiv b^{\frac{1}{3}}$. We can verify the inverse relationship between $\sqrt[3]{x}$ and x^3 by using either Property 1: $(\sqrt[3]{x})^3 = (x^{\frac{1}{3}})(x^{\frac{1}{3}})(x^{\frac{1}{3}}) = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x^1 = x$, or by using Property 4: $(\sqrt[3]{x})^3 = (x^{\frac{1}{3}})^3 = x^{\frac{3}{3}} = x^1 = x$.

Properties 5.1 and 5.2 also apply for fractional exponents:

$$\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}} = \sqrt[n]{a}\,\sqrt[n]{b}, \qquad \sqrt[n]{\left(\frac{a}{b}\right)} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Discussion

Even and odd exponents

The function $f(x) = x^n$ behaves differently depending on whether the exponent n is even or odd. If n is odd we have

$$\left(\sqrt[n]{b}\right)^n = \sqrt[n]{b^n} = b.$$

However, if n is even, the function x^n destroys the sign of the number (see x^2, which maps both $-x$ and x to x^2). The successive application of exponentiation by n and the n^{th} root has the same effect as the absolute value function:

$$\sqrt[n]{b^n} = |b|.$$

Recall that the absolute value function $|x|$ discards the information about the sign of x.

The expression $(\sqrt[n]{b})^n$ cannot be computed whenever b is a negative number. The reason is that we can't evaluate $\sqrt[n]{b}$ for $b < 0$ in terms of real numbers, since there is no real number which, multiplied times itself an even number of times, gives a negative number.

Scientific notation

In science we often work with very large numbers like *the speed of light* ($c = 299\,792\,458$[m/s]), and very small numbers like *the permeability of free space* ($\mu_0 = 0.000001256637\ldots$[N/A^2]). It can be difficult to judge the magnitude of such numbers and to carry out calculations on them using the usual decimal notation.

Dealing with such numbers is much easier if we use *scientific notation*. For example, the speed of light can be written as $c =$

2.99792458×10^8[m/s], and the permeability of free space is denoted as $\mu_0 = 1.256637 \times 10^{-6}$[N/A^2]. In both cases, we express the number as a decimal number between 1.0 and 9.9999... followed by the number 10 raised to some power. The effect of multiplying by 10^8 is to move the decimal point eight steps to the right, making the number bigger. Multiplying by 10^{-6} has the opposite effect, moving the decimal to the left by six steps and making the number smaller. Scientific notation is useful because it allows us to clearly see the *size* of numbers: 1.23×10^6 is 1 230 000 whereas 1.23×10^{-10} is 0.000 000 000 123. With scientific notation you don't have to count the zeros! Cool, yeah?

The number of decimal places we use when specifying a certain physical quantity is usually an indicator of the *precision* with which we are able to measure this quantity. Taking into account the precision of the measurements we make is an important aspect of all quantitative research. Since elaborating further would be a digression, we will not go into a full discussion about the topic of *significant digits* here. Feel free to check out the Wikipedia article on the subject if you want to know more.

On computer systems, *floating point numbers* are represented in scientific notation: they have a decimal part and an exponent. To separate the decimal part from the exponent when entering a floating point number into the computer, use the character `e`, which stands for "exponent." The base is assumed to be 10. For example, the speed of light is written as `2.99792458e8` and the permeability of free space is `1.256637e-6`.

Links

[Further reading on exponentiation]
`http://en.wikipedia.org/wiki/Exponentiation`

[More details on scientific notation]
`http://en.wikipedia.org/wiki/Scientific_notation`

1.9 Logarithms

Some people think the word "logarithm" refers to some mythical, mathematical beast. Legend has it that logarithms are many-headed, breathe fire, and are extremely difficult to understand. Nonsense! Logarithms are simple. It will take you at most a couple of pages to get used to manipulating them, and that is a good thing because logarithms are used all over the place.

The strength of your sound system is measured in logarithmic units called decibels [dB]. This is because your ears are sensitive only

1.13 Functions

We need to have a relationship talk. We need to talk about functions. We use functions to describe the relationships between variables. In particular, functions describe how one variable *depends* on another.

For example, the revenue R from a music concert depends on the number of tickets sold n. If each ticket costs \$25, the revenue from the concert can be written *as a function of* n as follows: $R(n) = 25n$. Solving for n in the equation $R(n) = 7000$ tells us the number of ticket sales needed to generate \$7000 in revenue. This is a simple model of a function; as your knowledge of functions builds, you'll learn how to build more detailed models of reality. For instance, if you need to include a 5% processing charge for issuing the tickets, you can update the revenue model to $R(n) = 0.95 \cdot 25 \cdot n$. If the estimated cost of hosting the concert is $C = \$2000$, then the profit from the concert P can be modelled as

$$\begin{aligned} P(n) &= R(n) \quad - \quad C \\ &= 0.95 \cdot \$25 \cdot n \quad - \quad \$2000 \end{aligned}$$

The function $P(n) = 23.75n - 2000$ models the profit from the concert as a function of the number of tickets sold. This is a pretty good model already, and you can always update it later on as you find out more information.

The more functions you know, the more tools you have for modelling reality. To "know" a function, you must be able to understand and connect several of its aspects. First you need to know the function's mathematical **definition**, which describes exactly what the function does. Starting from the function's definition, you can use your existing math skills to find the function's domain, its range, and its inverse function. You must also know the **graph** of the function; what the function looks like if you plot x versus $f(x)$ in the Cartesian plane (page 37). It's also a good idea to remember the **values** of the function for some important inputs. Finally—and this is the part that takes time—you must learn about the function's **relations** to other functions.

Definitions

A *function* is a mathematical object that takes numbers as inputs and gives numbers as outputs. We use the notation

$$f\colon A \to B$$

to denote a function from the input set A to the output set B. In this book, we mostly study functions that take real numbers as inputs and give real numbers as outputs: $f\colon \mathbb{R} \to \mathbb{R}$.

We now define some fancy technical terms used to describe the input and output sets.

- The *domain* of a function is the set of allowed input values.
- The *image* or *range* of the function f is the set of all possible output values of the function.
- The *codomain* of a function describes the type of outputs the function has.

To illustrate the subtle difference between the image of a function and its codomain, consider the function $f(x) = x^2$. The quadratic function is of the form $f\colon \mathbb{R} \to \mathbb{R}$. The function's domain is $\mathbb{R}$ (it takes real numbers as inputs) and its codomain is $\mathbb{R}$ (the outputs are real numbers too), however, not all outputs are possible. The *image* of the function $f(x) = x^2$ consists only of the nonnegative real numbers $[0, \infty \equiv \{y \in \mathbb{R} \mid y \geq 0\}$.

A function is not a number; rather, it is a *mapping* from numbers to numbers. For any input x, the output value of f for that input is denoted $f(x)$.

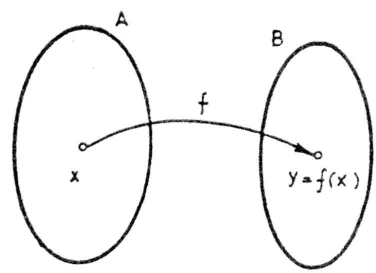


Figure 1.8: An abstract representation of a function f from the set A to the set B. The function f is the arrow which *maps* each input x in A to an output $f(x)$ in B. The output of the function $f(x)$ is also denoted y.

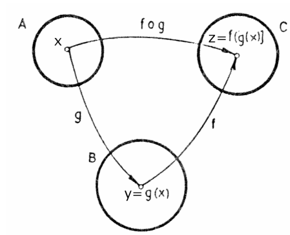
We say "f maps x to $f(x)$," and use the following terminology to classify the type of mapping that a function performs:

- A function is *one-to-one* or *injective* if it maps different inputs to different outputs.
- A function is *onto* or *surjective* if it covers the entire output set (in other words, if the image of the function is equal to the function's codomain).
- A function is *bijective* if it is both injective and surjective. In this case, f is a *one-to-one correspondence* between the input set and the output set: for each of the possible outputs $y \in Y$ (surjective part), there exists exactly one input $x \in X$, such that $f(x) = y$ (injective part).

The term *injective* is an allusion from the 1940s inviting us to picture the actions of injective functions as pipes through which numbers flow like fluids. Since a fluid cannot be compressed, the output space must be at least as large as the input space. A modern synonym for injective functions is to say they are *two-to-two*. If we imagine two specks of paint floating around in the "input fluid," an injective function will contain two distinct specks of paint in the "output fluid." In contrast, non-injective functions can map several different inputs to the same output. For example $f(x) = x^2$ is not injective since the inputs 2 and -2 are both mapped to the output value 4.

Function composition

We can combine two simple functions by chaining them together to build a more complicated function. This act of applying one function after another is called *function composition*. Consider for example the composition:

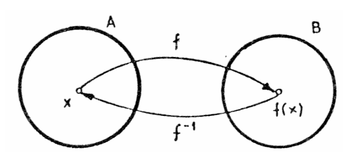


$$f\circ g\,(x) \equiv f(\,g(x)\,) = z.$$

The diagram on the right illustrates what is going on. First, the function $g : A \to B$ acts on some input x to produce an intermediary value $y = g(x)$ in the set B. The intermediary value y is then passed through the function $f : B \to C$ to produce the final output value $z = f(y) = f(g(x))$ in the set C. We can think of the *composite function* $f \circ g$ as a function in its own right. The function $f \circ g : A \to C$ is defined through the formula $f \circ g\,(x) \equiv f(g(x))$.

Inverse function

Recall that a *bijective* function is a one-to-one correspondence between a set of input values and a set of output values. Given a bijective function $f : A \to B$, there exists an inverse function $f^{-1} : B \to A$, which performs the *inverse mapping* of f. If you start from some x, apply f, and then apply f^{-1}, you'll arrive—full circle—back to the original input x:



$$f^{-1}(\,f(x)\,) \equiv f^{-1}\circ f\;(x) = x.$$

This inverse function is represented abstractly as a backward arrow, that puts the value $f(x)$ back to the x it came from.

Square root

The square root function is denoted

$$f(x) = \sqrt{x} \equiv x^{\frac{1}{2}}.$$

The square root $\sqrt{x}$ is the inverse function of the square function x^2 for $x \geq 0$. The symbol $\sqrt{c}$ refers to the *positive* solution of $x^2 = c$. Note that $-\sqrt{c}$ is also a solution of $x^2 = c$.

Graph

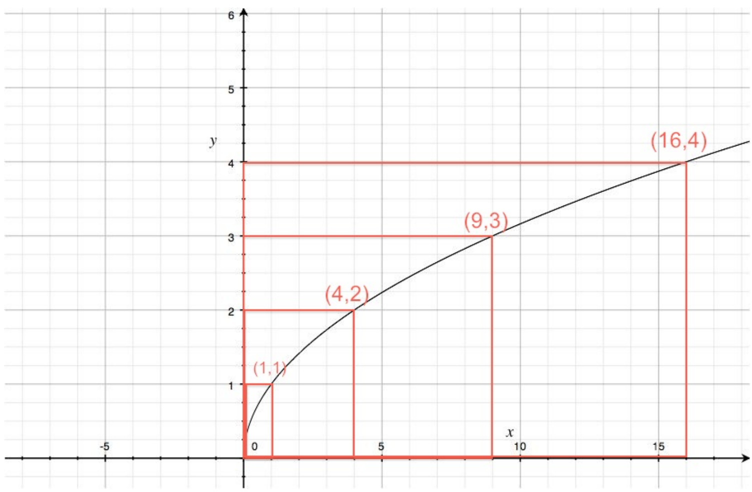


Figure 1.12: The graph of the function $f(x) = \sqrt{x}$. The domain of the function is $x \in [0, \infty$. You can't take the square root of a negative number.

Properties

- Domain: $x \in [0, \infty$.
  The function $f(x) = \sqrt{x}$ is only defined for nonnegative inputs $x \geq 0$. There is no real number y such that y^2 is negative, hence the function $f(x) = \sqrt{x}$ is not defined for negative inputs x.
- Image: $f(x) \in [0, \infty$.
  The outputs of the function $f(x) = \sqrt{x}$ are never negative: $\sqrt{x} \geq 0$, for all $x \in [0, \infty$.

In addition to *square* root, there is also *cube* root $f(x) = \sqrt[3]{x} \equiv x^{\frac{1}{3}}$, which is the inverse function for the cubic function $f(x) = x^3$. We have $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 8$. More generally, we can define the n^{th}-root function $\sqrt[n]{x}$ as the inverse function of x^n.

Natural logarithm

The natural logarithm function is denoted

$$f(x) = \ln(x) = \log_e(x).$$

The function $\ln(x)$ is the inverse function of the exponential e^x.

Graph

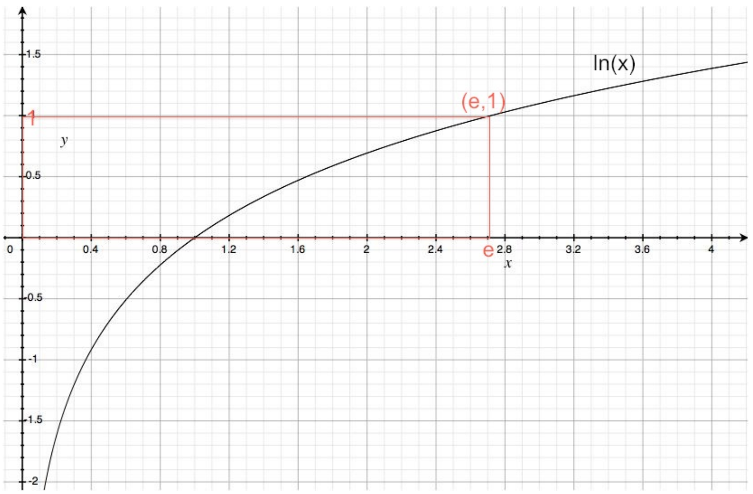


Figure 1.19: The graph of the function $\ln(x)$ passes through the following (x,y) coordinates: $(\frac{1}{e^2}, -2)$, $(\frac{1}{e}, -1)$, $(1,0)$, $(e,1)$, $(e^2,2)$, $(e^3,3)$, $(148.41\ldots,5)$, and $(22026.46\ldots,10)$.

Function transformations

Often, we're asked to adjust the shape of a function by scaling it or moving it, so that it passes through certain points. For example, if we wanted to make a function g with the same shape as the absolute value function $f(x) = |x|$, but for which $g(0) = 3$, we would use the function $g(x) = |x| + 3$.

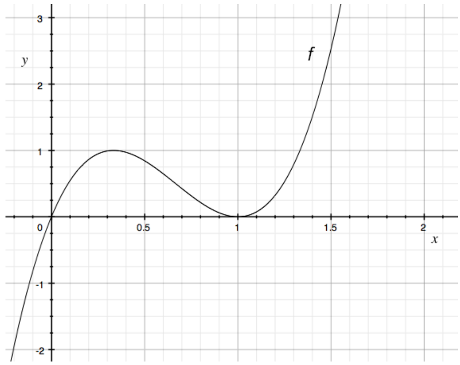
In this section, we'll discuss the four basic transformations you can perform on *any* function f to obtain a transformed function g:

- Vertical translation: $g(x) = f(x) + k$
- Horizontal translation: $g(x) = f(x - h)$
- Vertical scaling: $g(x) = Af(x)$
- Horizontal scaling: $g(x) = f(ax)$

By applying these transformations, we can *move* and *stretch* a generic function to give it any desired shape.

The next couple of pages illustrate all of the above transformations on the function

$$f(x) = 6.75(x^3 - 2x^2 + x).$$



We'll work with this function because it has distinctive features in both the horizontal and vertical directions. By observing this function's graph, we see its x-intercepts are at $x = 0$ and $x = 1$. We can confirm this mathematically by factoring the expression:

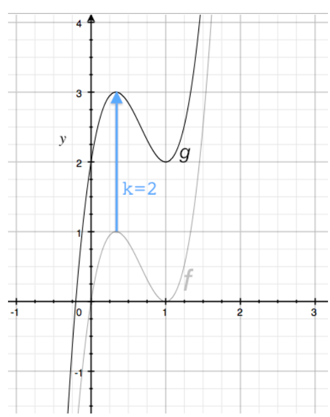
$$f(x) = 6.75x(x^2 - 2x + 1) = 6.75x(x-1)^2.$$

The function $f(x)$ also has a local maximum at $x = \frac{1}{3}$, and the value of the function at that maximum is $f(\frac{1}{3}) = 1$.

Vertical translations

To move a function $f(x)$ *up* by k units, add k to the function:

$$g(x) = f(x) + k.$$



The function $g(x)$ will have exactly the same shape as $f(x)$, but it will be *translated* (the mathematical term for moved) upward by k units.

Recall the function $f(x) = 6.75(x^3 - 2x^2 + x)$. To move the function up by $k = 2$ units, we can write

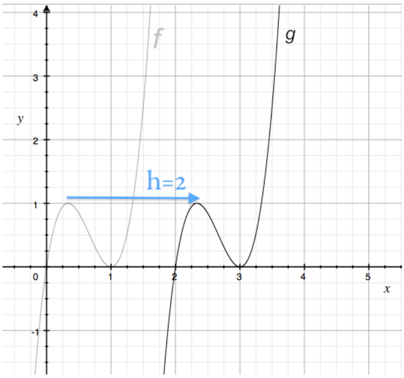
$$g(x) = f(x)+2 = 6.75(x^3-2x^2+x)+2,$$

and the graph of $g(x)$ will be as it is shown to the right. Recall the original function $f(x)$ crosses the x-axis at $x = 0$. The transformed function $g(x)$ has the property $g(0) = 2$. The maximum at $x = \frac{1}{3}$ has similarly shifted in value from $f(\frac{1}{3}) = 1$ to $g(\frac{1}{3}) = 3$.

Horizontal translation

We can move a function f to the right by h units by *subtracting* h from x and using $(x-h)$ as the function's input argument:

$$g(x) = f(x-h).$$



The point $(0, f(0))$ on $f(x)$ now corresponds to the point $(h, g(h))$ on $g(x)$.

The graph to the right shows the function $f(x) = 6.75(x^3 - 2x^2 + x)$, as well as the function $g(x)$, which is shifted to the right by $h = 2$ units:

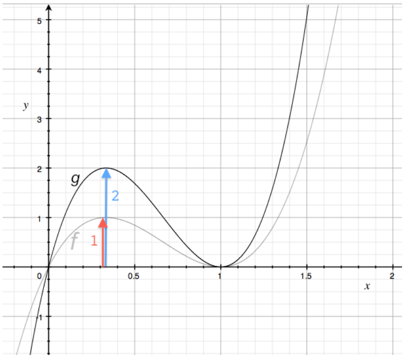
$$g(x) = f(x-2) = 6.75\left[(x-2)^3 - 2(x-2)^2 + (x-2)\right].$$

The original function f gives us $f(0) = 0$ and $f(1) = 0$, so the new function $g(x)$ must give $g(2) = 0$ and $g(3) = 0$. The maximum at $x = \frac{1}{3}$ has similarly shifted by two units to the right, $g(2 + \frac{1}{3}) = 1$.

Vertical scaling

To stretch or compress the shape of a function vertically, we can multiply it by some constant A and obtain

$$g(x) = Af(x).$$



If $|A| > 1$, the function will be stretched. If $|A| < 1$, the function will be compressed. If A is negative, the function will flip upside down, which is a *reflection* through the x-axis.

There is an important difference between vertical translation and vertical scaling. Translation moves all points of the function by the same amount, whereas scaling moves each point proportionally to that point's distance from the x-axis.

The function $f(x) = 6.75(x^3 - 2x^2 + x)$, when stretched vertically by a factor of $A = 2$, becomes the function

$$g(x) = 2f(x) = 13.5(x^3 - 2x^2 + x).$$

The x-intercepts $f(0) = 0$ and $f(1) = 0$ do not move, and remain at $g(0) = 0$ and $g(1) = 0$. The maximum at $x = \frac{1}{3}$ has doubled in value as $g(\frac{1}{3}) = 2$. Indeed, all values of $f(x)$ have been stretched upward by a factor of 2, as we can verify using the point $f(1.5) = 2.5$, which has become $g(1.5) = 5$.

Horizontal scaling

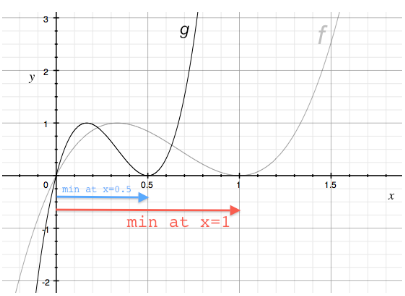
To stretch or compress a function horizontally, we can multiply the input value by some constant a to obtain:

$$g(x) = f(ax).$$

If $|a| > 1$, the function will be compressed. If $|a| < 1$, the function will be stretched. Note that the behaviour here is the opposite of vertical scaling. If a is a negative number, the function will also flip horizontally, which is a reflection through the y-axis.

The graph on the right shows $f(x) = 6.75(x^3 - 2x^2 + x)$, as well as the function $g(x)$, which is $f(x)$ compressed horizontally by a factor of $a = 2$:

$$\begin{aligned} g(x) &= f(2x) \\ &= 6.75\big[(2x)^3 - 2(2x)^2 + (2x)\big]. \end{aligned}$$



The x-intercept $f(0) = 0$ does not move since it is on the y-axis. The x-intercept $f(1) = 0$ does move, however, and we have $g(0.5) = 0$. The maximum at $x = \frac{1}{3}$ moves to $g(\frac{1}{6}) = 1$. All values of $f(x)$ are compressed toward the y-axis by a factor of 2.

General quadratic function

The general quadratic function takes the form

$$f(x) = A(x - h)^2 + k,$$

where x is the input, and $A, h,$ and k are the *parameters*.

1.16 Trigonometry

We can put any three lines together to make a triangle. What's more, if one of the triangle's angles is equal to 90°, we call this triangle a *right-angle triangle*.

In this section we'll discuss right-angle triangles in great detail and get to know their properties. We'll learn some fancy new terms like *hypotenuse*, *opposite*, and *adjacent*, which are used to refer to the different sides of a triangle. We'll also use the functions *sine*, *cosine*, and *tangent* to compute the *ratios of lengths* in right triangles.

Understanding triangles and their associated trigonometric functions is of fundamental importance: you'll need this knowledge for your future understanding of mathematical subjects like vectors and complex numbers, as well as physics subjects like oscillations and waves.

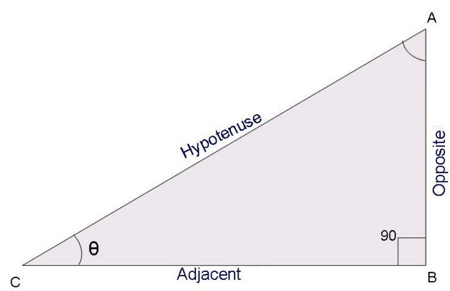


Figure 1.21: A right-angle triangle. The angle θ and the names of the sides of the triangle are indicated.

Concepts

- A, B, C: the three *vertices* of the triangle
- θ: the angle at the vertex C. Angles can be measured in degrees or radians.
- $\text{opp} \equiv \overline{AB}$: the length of the *opposite* side to θ
- $\text{adj} \equiv \overline{BC}$: the length of side *adjacent* to θ
- $\text{hyp} \equiv \overline{AC}$: the *hypotenuse*. This is the triangle's longest side.
- h: the "height" of the triangle (in this case $h = \text{opp} = \overline{AB}$)
- $\sin\theta \equiv \frac{\text{opp}}{\text{hyp}}$: the *sine* of theta is the ratio of the length of the opposite side and the length of hypotenuse.
- $\cos\theta \equiv \frac{\text{adj}}{\text{hyp}}$: the *cosine* of theta is the ratio of the adjacent length and the hypotenuse length.
- $\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \equiv \frac{\text{opp}}{\text{adj}}$: the *tangent* is the ratio of the opposite length divided by the adjacent length.

1.25 Math problems

We've now reached the first section of problems in this book. The purpose of these problems is to give you a way to comprehensively practice your math fundamentals. In the real world, you'll rarely have to solve equations by hand, however, knowing how to solve math equations and manipulate math expressions will be very useful in later chapters of this book. At times, honing your math chops might seem like tough mental work, but at the end of each problem, you'll gain a stronger foothold on all the subjects you've been learning about. You'll also experience a small *achievement buzz* after each problem you vanquish.

I have a special message to readers who are learning math just for fun: you can either try the problems in this section or skip them. Since you have no upcoming exam on this material, you could skip ahead to Chapter 2 without any immediate consequences. However (and it's a big however), those readers who don't take a crack at these problems will be missing a significant opportunity.

Sit down to do them later today, or another time when you're properly caffeinated. If you take the initiative to make time for math, you'll find yourself developing lasting comprehension and true math fluency. Without the practice of solving problems, however, you're extremely likely to forget most of what you've learned within a month, simple as that. You'll still remember the big ideas, but the details will be fuzzy and faded. Don't break the pace now: with math, it's very much *use it or lose it*!

By solving some of the problems in this section, you'll remember a lot more stuff. Make sure you step away from the pixels while you're solving problems. You don't need fancy technology to do math; grab a pen and some paper from the printer and you'll be fine. Do yourself a favour: put your phone in airplane-mode, close the lid of your laptop, and move away from desktop computers. Give yourself some time to think. Yes, I know you can look up the answer to any question in five seconds on the Internet, and you can use `live.sympy.org` to solve any math problem, but that is like outsourcing the thinking. Descartes, Leibniz, and Riemann did most of their work with pen and paper and they did well. Spend some time with math the way the masters did.

P1.1 Solve for x in the equation $x^2 - 9 = 7$.

P1.2 Solve for x in the equation $\cos^{-1}\left(\frac{x}{A}\right) - \phi = \omega t$.

P1.3 Solve for x in the equation $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$.

P1.4 Use a calculator to find the values of the following expressions:

(1) $\sqrt[4]{3^3}$ (2) 2^{10} (3) $7^{\frac{1}{4}} - 10$ (4) $\frac{1}{2}\ln(e^{22})$

P1.5 Compute the following expressions involving fractions:

(1) $\dfrac{1}{2} + \dfrac{1}{4}$ (2) $\dfrac{4}{7} - \dfrac{23}{5}$ (3) $1\frac{3}{4} + 1\frac{31}{32}$

P1.6 Use the basic rules of algebra to simplify the following expressions:

(1) $ab\,\dfrac{1}{a}\,b^2cb^{-3}$ (2) $\dfrac{abc}{bca}$ (3) $\dfrac{27a^2}{\sqrt{9abba}}$

(4) $\dfrac{a(b+c)-ca}{b}$ (5) $\dfrac{a}{c\sqrt[3]{b}}\,\dfrac{b^{\frac{4}{3}}}{a^2}$ (6) $(x+a)(x+b) - x(a+b)$

P1.7 Expand the brackets in the following expressions:

(1) $(x+a)(x-b)$ (2) $(2x+3)(x-5)$ (3) $(5x-2)(2x+7)$

P1.8 Factor the following expressions as a product of linear terms:

(1) $x^2 - 2x - 8$ (2) $3x^3 - 27x$ (3) $6x^2 + 11x - 21$

P1.9 Complete the square in the following quadratic expressions to obtain expressions of the form $A(x-h)^2 + k$.

(1) $x^2 - 4x + 7$ (2) $2x^2 + 12x + 22$ (3) $6x^2 + 11x - 21$

P1.10 A golf club and a golf ball cost \$1.10 together. The golf club costs one dollar more than the ball. How much does the ball cost?

P1.11 An ancient artist drew scenes of hunting on the walls of a cave, including 43 figures of animals and people. There were 17 more figures of animals than people. How many figures of people did the artist draw and how many figures of animals?

P1.12 A father is 35 years old and his son is 5 years old. In how many years will the father's age be four times the son's age?

P1.13 A boy and a girl collected 120 nuts. The girl collected twice as many nuts as the boy. How many nuts did each collect?

P1.14 Alice is 5 years older than Bob. The sum of their ages is 25 years. How old is Alice?

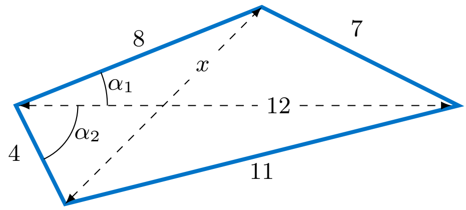
P1.15 A publisher needs to bind 4500 books. One print shop can bind these books in 30 days, another shop can do it in 45 days. How many days are necessary to bind all the books if both shops work in parallel?
Hint: Find the books-per-day rate of each shop.

P1.16 A plane leaves Vancouver travelling at 600 km/h toward Montreal. One hour later, a second plane leaves Vancouver heading for Montreal at 900 km/h. How long will it take for the second plane to overtake the first?
Hint: Distance travelled is equal to velocity multiplied by time: $d = vt$.

P1.17 There are 26 sheep and 10 goats on a ship. How old is the captain?

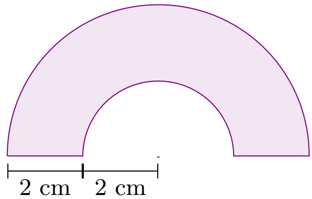
P1.18 The golden ratio, denoted φ, is the positive solution to the quadratic equation $x^2 - x - 1 = 0$. Find the golden ratio.

P1.42 Find the length x of the diagonal of the quadrilateral below.



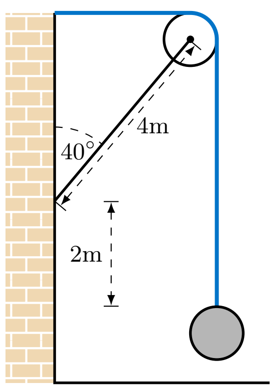
Hint: Use the law of cosines once to find α_1 and α_2, and again to find x.

P1.43 Find the area of the shaded region.



Hint: Find the area of the outer circle, subtract the area of missing centre disk, then divide by two.

P1.44 In preparation for the shooting of a music video, you're asked to suspend a wrecking ball hanging from a circular pulley. The pulley has a radius of 50 cm. The other lengths are indicated in the figure. What is the total length of the rope required?



Hint: The total length of rope consists of two straight parts and the curved section that wraps around the pulley.

P1.45 The length of a rectangle is $c + 2$ and its height is 5. What is the area of the rectangle?

P1.46 A box of facial tissues has dimensions 10.5 cm by 7 cm by 22.3 cm. What is the volume of the box in litres?
Hint: $1\,\text{L} = 1000\,\text{cm}^3$.

Chapter 2

Introduction to physics

2.1 Introduction

One of the coolest things about understanding math is that you will automatically start to understand the laws of physics too. Indeed, most physics laws are expressed as mathematical equations. If you know how to manipulate equations and you know how to solve for the unknowns in them, then you know half of physics already.

Ever since Newton figured out the whole $F = ma$ thing, people have used mechanics to achieve great technological feats, like landing spaceships on the Moon and Mars. You can be part of this science thing too. Learning physics will give you the following superpowers:

1. The power to **predict the future motion of objects** using equations. For most types of motion, it is possible to find an equation that describes the position of an object as a function of time $x(t)$. You can use this equation to predict the position of the object at all times t, including the future. "Yo G! Where's the particle going to be at $t = 1.3$ seconds?" you are asked. "It is going to be at $x(1.3)$ metres, bro." Simple as that. The equation $x(t)$ describes the object's position for *all* times t during the motion. Knowing this, you can plug $t = 1.3$ seconds into $x(t)$ to find the object's location at that time.

2. Special **physics vision** for seeing the world. After learning physics, you will start to think in term of concepts like force, acceleration, and velocity. You can use these concepts to precisely describe all aspects of the motion of objects. Without physics vision, when you throw a ball into the air you will see it go up, reach the top, then fall down. Not very exciting. Now *with* physics vision, you will see that at $t = 0$[s], the same

ball is thrown in the positive y-direction with an initial velocity of $v_i = 12[\text{m/s}]$. The ball reaches a maximum height of $\max\{y(t)\} = \frac{12^2}{2\times 9.81} = 7.3[\text{m}]$ at $t = 12/9.81 = 1.22[\text{s}]$, then hits the ground after a total flight time of $t_f = 2\sqrt{\frac{2\times 7.3}{9.81}} = 2.44[\text{s}]$.

The *measurement units* of physical quantities throughout this book are denoted in square brackets, like in the example above. Learning about the different measurement units is an important aspect of *physics vision*.

Why learn physics?

The main reason why you should learn physics is for the *knowledge buzz*. You will learn how to calculate the motion of objects, predict the outcomes of collisions, describe oscillations, and many other useful things. As you develop your physics skills, you will be able to use physics equations to derive one physical quantity from another. For example, we can predict the maximum height reached by a ball, if we know its initial velocity when thrown. The equations of physics are a lot like LEGOs; your job is to figure out different ways to connect them together.

By learning how to solve complicated physics problems, you will develop your analytical skills. Later on, you can apply these skills to other areas of life. Even if you don't go on to study science, the expertise you develop in solving physics problems will help you tackle complicated problems in general. As proof of this statement, consider the fact that companies like to hire physicists even for positions unrelated to physics: they feel confident that candidates who understand physics will be able to figure out all the business stuff easily.

Intro to science

Perhaps the most important reason you should learn physics is because it represents the golden standard for the scientific method. First of all, physics deals only with concrete things that can be **measured**. There are no feelings or subjectivities in physics. Physicists must derive mathematical models that **accurately describe** and **predict** the outcomes of experiments. Above all, we can **test** the validity of the physical models by running experiments and comparing the predicted outcome with what actually happens in the lab.

The key ingredient in scientific thinking is skepticism. Scientists must convince their peers that their equations are true without a doubt. The peers shouldn't need to *trust* the scientist; rather, they

can carry out their own tests to see if the equation accurately predicts what happens in the real world. For example, let's say I claim that the height of a ball thrown up in the air with speed 12[m/s] is described by the equation $y_c(t) = \frac{1}{2}(-9.81)t^2 + 12t + 0$. To test whether this equation is true, you can perform a throwing-the-ball-in-the-air experiment and record the motion of the ball as a video. You can then compare the motion parameters observed in the video with those predicted by the claimed equation $y_c(t)$.

- **Maximum height reached** One thing you can check is whether the equation $y_c(t)$ predicts the ball's maximum height $y_{\max}$. The claimed equation predicts the ball will reach its maximum height at $t = 1.22$[s]. The maximum height predicted is $\max_t\{y_c(t)\} = y_c(1.22) = 7.3$[m]. You can compare this value with the maximum height $y_{\max}$ you observe in the video.

- **Total time of flight** You can also check whether the equation $y_c(t)$ correctly predicts the time when the ball will fall back to the ground. Using the video, suppose you measure the time it took the ball to fall back to the ground to be $t_{\text{fall}} = 2.44$[s]. If the equation $y_c(t)$ is correct, it should predict a height of zero metres for the time t_{fall}.

If both predictions of the equation $y_c(t)$ match your observations from the video, you can start to believe the claimed equation of motion $y_c(t)$ is truly an accurate model for the real world.

The scientific method depends on this interplay between experiment and theory. Theoreticians prove theorems and derive equations, while experimentalists test the validity of equations. The equations that accurately predict the laws of nature are kept while inaccurate models are rejected. At the same time, experimentalists constantly measure new data and challenge theoreticians to come up with equations that correctly describe the new measurements.

Equations of physics

The best physics equations are collected in textbooks. Physics textbooks contain only equations that have been extensively tested and are believed to be true. Good physics textbooks also explain how the equations are *derived* from first principles. This is important, because it is much easier to understand a few fundamental principles of physics, rather than memorize a long list of formulas. Understanding trumps memorization any day of the week.

The next section will teach you about three equations that fully describe the motion of any object: $x(t)$, $v(t)$, and $a(t)$. Using these equations and the equation-solving techniques from Chapter 1, we can

predict pretty much anything we want about the position and velocity of objects undergoing *constant acceleration*.
Instead of asking you to memorize these equations, I'll show you a cool trick for obtaining one equation of motion from another. These three equations describe different aspects of the same motion, so it's no surprise the equations are related. While you are not required to know how to derive the equations of physics, you do need to know how to use all these equations. Learning a bit of theory is a good deal: just a few pages of "difficult" theory (integrals) will give you a deep understanding of the relationship between $a(t)$, $v(t)$, and $x(t)$. This way, you can rely on your newly expanded math knowledge, rather than remember three separate formulas!

2.2 Kinematics

Kinematics (from the Greek word *kinema* for *motion*) is the study of trajectories of moving objects. The equations of kinematics can be used to calculate how long a ball thrown upward will stay in the air, or to calculate the acceleration needed to go from 0 to 100[km/h] in 5 seconds. To carry out these calculations, we need to choose the right *equation of motion* and figure out the values of the *initial conditions* (the initial position x_i and the initial velocity v_i). Afterward, we plug the known values into the appropriate equation of motion and solve for the unknown using one or two simple algebra steps. This entire section boils down to three equations and the plug-number-into-equation skill.

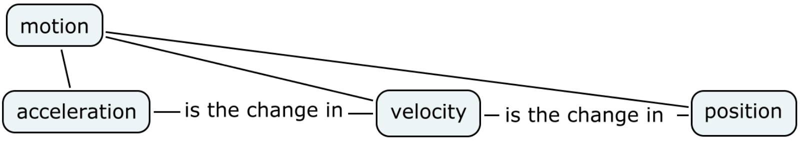


Figure 2.1: The motion of an object is described by its position, velocity, and acceleration functions.

This section is here to teach you how to use the equations of motion and help you understand the concepts of velocity and acceleration. You'll also learn how to recognize which equations to use when solving different types of physics problems.

Concepts

The key notions for describing the motion of objects are:

- t: the time. Time is measured in seconds [s].
- $x(t)$: an object's position as a function of time—also known as the equation of motion. Position is measured in metres [m] and depends on the time t.
- $v(t)$: the object's velocity as a function of time. Velocity is measured in metres per second [m/s].
- $a(t)$: the object's acceleration as a function of time. Acceleration is measured in metres per second squared [m/s^2].
- $x_i = x(0), v_i = v(0)$: the object's initial position and velocity, as measured at $t = 0$. Together x_i and v_i are known as the *initial conditions*.

Position, velocity, and acceleration

The motion of an object is characterized by three functions: the position function $x(t)$, the velocity function $v(t)$, and the acceleration function $a(t)$. The functions $x(t)$, $v(t)$, and $a(t)$ are connected—they all describe different aspects of the same motion.

You are already familiar with these notions from your experience of riding in a car. The equation of motion $x(t)$ describes the position of the car as a function of time. The velocity describes the change in the position of the car, or mathematically,

$$v(t) \equiv \text{rate of change in } x(t).$$

If we measure x in metres [m] and time t in seconds [s], then the units of $v(t)$ will be metres per second [m/s]. For example, an object moving with a constant velocity of +30[m/s] will increase its position by 30[m] each second. Note that the velocity $v(t)$ could be positive or negative. The *speed* of an object is defined as the absolute value of its velocity $|v(t)|$.

The rate of change of an object's velocity is called *acceleration*:

$$a(t) \equiv \text{rate of change in } v(t).$$

Acceleration is measured in metres per second squared [m/s^2]. A constant positive acceleration means the velocity of the motion is steadily increasing, similar to pressing the gas pedal. A constant negative acceleration means the velocity is steadily decreasing, similar to pressing the brake pedal.

In a couple of paragraphs, we'll discuss the exact mathematical equations for $x(t)$, $v(t)$, and $a(t)$, but before we dig into the math, let's look at the example of the motion of a car illustrated in Figure 2.2.

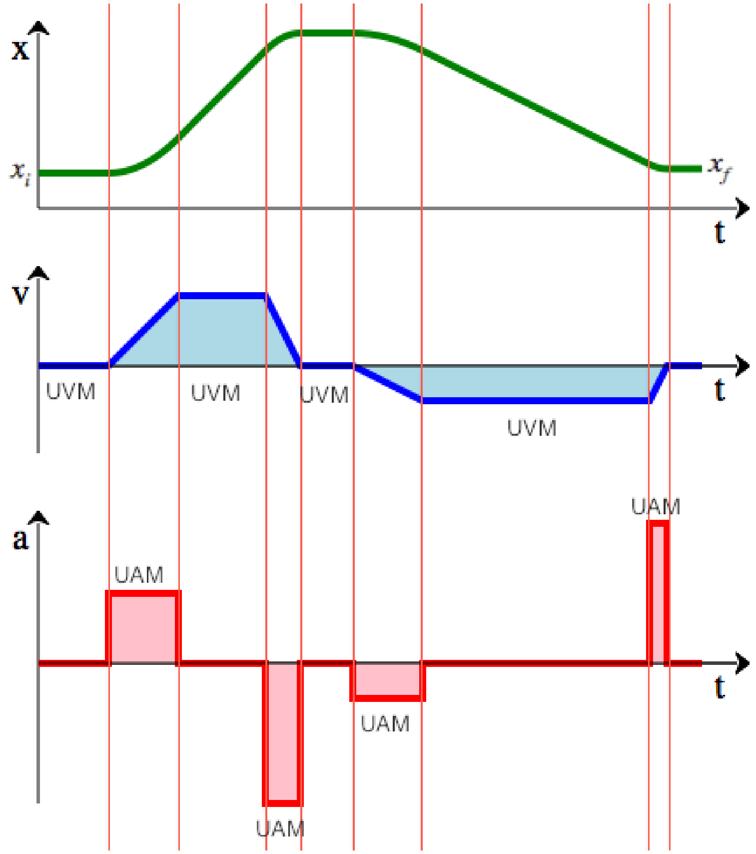


Figure 2.2: The illustration shows the simultaneous graphs of the position, velocity, and acceleration of a car during some time interval. The car starts from an initial position x_i where it sits still for some time. The driver then floors the pedal to produce a maximum acceleration for some time, and the car picks up speed. The driver then eases off the accelerator, keeping it pressed enough to maintain a constant speed. Suddenly the driver sees a police vehicle in the distance and slams on the brakes (negative acceleration) and shortly afterward brings the car to a stop. The driver waits for a few seconds to make sure the cops have passed. Next, the driver switches into reverse gear and adds gas. The car accelerates backward for a bit, then maintains a constant backward speed for an extended period of time. Note how "moving backward" corresponds to negative velocity. In the end the driver slams on the brakes again to stop the car. Notice that braking corresponds to positive acceleration when the motion is in the negative direction. The car's final position is x_f.

We can observe two distinct types of motion in the situation described in Figure 2.2. During some times, the car undergoes motion at a constant velocity (uniform velocity motion, UVM). During other times, the car undergoes motion with constant acceleration (uniform acceleration motion, UAM). There exist many other types of motion, but for the purpose of this section we'll focus on these two types of motion.

- UVM: During times when there is no acceleration, the car maintains a uniform velocity and therefore $v(t)$ is a constant function. For motion with constant velocity, the position function is a line with a constant slope because, by definition, $v(t) =$ slope of $x(t)$.
- UAM: During times where the car experiences a constant acceleration $a(t) = a$, the velocity of the function changes at a constant rate. The rate of change of the velocity is constant $a =$ slope of $v(t)$, so the velocity function looks like a line with slope a. The position function $x(t)$ has a curved shape (quadratic) during moments of constant acceleration.

Formulas

There are basically four equations you need to know for this entire section. Together, these four equations fully describe all aspects of motion with constant acceleration.

Uniformly accelerated motion (UAM)

If the object undergoes a *constant* acceleration $a(t) = a$—like a car when you floor the *accelerator*—then its motion can be described by the following equations:

$$a(t) = a, \tag{2.1}$$

$$v(t) = at + v_i, \tag{2.2}$$

$$x(t) = \tfrac{1}{2}at^2 + v_i t + x_i, \tag{2.3}$$

where v_i is the initial velocity of the object and x_i is its initial position.

Here is another useful equation to remember:

$$[v(t)]^2 = v_i^2 + 2a[x(t) - x_i],$$

which is usually written

$$v_f^2 = v_i^2 + 2a\Delta x, \tag{2.4}$$

where v_f denotes the final velocity (at $t = t_f$) and Δx denotes the *change* in the x-coordinate between $t = 0$ and $t = t_f$. The triangle

thing Δ is the capital Greek letter *delta*, which is often used to denote the change in quantities. Using the Δ-notation, we can rewrite equation (2.2) as follows: $\Delta v = a\Delta t$, where $\Delta v \equiv v_f - v_i$ and $\Delta t \equiv t_f - t_i$.

That's it! Memorize these four equations, plug-in the right numbers, and you can solve any kinematics problem humanly imaginable.

Uniform velocity motion (UVM)

The special case where there is zero acceleration ($a = 0$), is called *uniform velocity motion* or UVM. The velocity stays uniform (constant) because there is no acceleration. The following three equations describe the motion of an object with uniform velocity:

$$\begin{aligned} a(t) &= 0, \\ v(t) &= v_i, \\ x(t) &= v_i t + x_i. \end{aligned}$$

As you can see, these are really the same equations as in the UAM case above, but because $a = 0$, some terms are missing.

Free fall

We say an object is in *free fall* if the only force acting on it is the force of gravity. On the surface of the Earth, the force of gravity produces a constant acceleration of $a_y = -9.81[\text{m/s}^2]$. The negative sign is there because the gravitational acceleration is directed downward, and we assume the y-axis points upward. Since the gravitational acceleration is constant, we can use the UAM equations to find the height $y(t)$ and velocity $v(t)$ of objects in free fall.

Examples

Now we'll illustrate how the equations of kinematics are used.

Moroccan example Suppose your friend wants to send you a ball wrapped in aluminum foil by dropping it from his balcony, which is located at a height of $y_i = 44.145[\text{m}]$. How long will it take for the ball to hit the ground?

We recognize this is a problem with acceleration, so we start by writing the general UAM equations:

$$\begin{aligned} y(t) &= \tfrac{1}{2}at^2 + v_i t + y_i, \\ v(t) &= at + v_i. \end{aligned}$$

To find the answer, substitute the following known values into the $y(t)$ equation: $y(0) = y_i = 44.145[\text{m}]$; $a = -9.81$ (since the ball is in

free fall); and $v_i = 0$[m/s] (since the ball was released from rest). We want to find the time t_{fall} when the height of the ball will be zero:

$$\begin{aligned} 0 &= y(t_{\text{fall}}), \\ 0 &= \tfrac{1}{2}(-9.81)(t_{\text{fall}})^2 + 0(t_{\text{fall}}) + 44.145. \end{aligned}$$

Solving for t_{fall} we find the answer $t_{\text{fall}} = \sqrt{\frac{44.145\times 2}{9.81}} = 3[\text{s}]$.

As another variation of this type of kinematics question, suppose you're given the time it takes for the ball to fall $t_{\text{fall}} = 3$[s], and you're asked to find the height of the balcony. You already know $y(3) = 0$, and are looking for the initial height y_i. You can solve for y_i in the equation $0 = \frac{1}{2}(-9.81)3^2 + y_i$. The answer gives $y_i = 44.145$[m].

0 to 100 in 5 seconds Say you're in the driver's seat of a car and you want to accelerate from 0 to 100[km/h] in 5 seconds. How much acceleration must the car's engine produce, assuming it produces a constant amount of acceleration?

We can calculate the necessary a by plugging the required values into the velocity equation for UAM:

$$v(t) = at + v_i.$$

Before we tackle that, we need to convert the velocity in [km/h] to velocity in [m/s]: $100[\text{km/h}] = \frac{100[\cancel{\text{km}}]}{1[\cancel{\text{h}}]} \cdot \frac{1000[\text{m}]}{1[\cancel{\text{km}}]} \cdot \frac{1[\cancel{\text{h}}]}{3600[\text{s}]} = 27.8\ [\text{m/s}]$. We substitute the desired values $v_f = 27.8$[m/s], $v_i = 0$, and $t = 5$[s] into the equation for $v(t)$ and solve for a:

$$27.8 = v(5) = a5 + 0.$$

After solving for a, we find the car's engine must produce a constant acceleration of $a = \frac{27.8}{5} = 5.56[\text{m/s}^2]$ or greater.

Moroccan example II Some time later, your friend wants to send you another aluminum ball from his apartment located on the 14^{th} floor (height of 44.145[m]). To decrease the time of flight, he *throws* the ball straight down with an initial velocity of 10[m/s]. How long does it take for the ball to hit the ground?

Imagine the apartment building as a y-axis that measures distance upward starting from the ground floor. We know the balcony is located at a height of $y_i = 44.145$[m], and that at $t = 0$[s] the ball starts with $v_i = -10$[m/s]. The initial velocity is negative because it points in the opposite direction of the y-axis. We also know there is an acceleration due to gravity of $a_y = -9.81[\text{m/s}^2]$.

We start by writing the general UAM equation:

$$y(t) = \tfrac{1}{2}a_y t^2 + v_i t + y_i.$$

To find the time when the ball will hit the ground, we must solve for t in the equation $y(t) = 0$. Plug all the known values into the UAM equation,

$$y(t) = 0 = \tfrac{1}{2}(-9.81)t^2 - 10t + 44.145,$$

and solve for t using the quadratic formula. First, rewrite the quadratic equation in standard form:

$$0 = \underbrace{4.905}_{a} t^2 + \underbrace{10.0}_{b} t \quad \underbrace{-44.145}_{c}.$$

Then solve using the quadratic equation:

$$t_{\text{fall}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10 \pm \sqrt{100 + 866.12}}{9.81} = 2.15 \qquad [\text{s}].$$

We ignore the negative-time solution because it corresponds to a time in the past. Compared to the first Moroccan example, we see that throwing the ball downward makes it fall to the ground faster.

Discussion

Most kinematics problems you'll be solving follow the same pattern as the examples above. Given some initial values, you'll be asked to solve for some unknown quantity.

It's important to keep in mind the *signs* of the numbers you plug into the equations. You should always draw the coordinate system and indicate clearly (to yourself) the x-axis, which measures the object's displacement. A velocity or acceleration quantity that points in the same direction as the x-axis is a positive number, while quantities pointing in the opposite direction are negative numbers.

By the way, all this talk about $v(t)$ being the "rate of change of $x(t)$" is starting to get on my nerves. The expression "rate of change of" is an indirect way of saying the calculus term *derivative*. In order to use this more precise terminology throughout the remainder of the book, we'll now take a short excursion into the land of calculus to define two fundamental concepts: derivatives and integrals.

2.3 Introduction to calculus

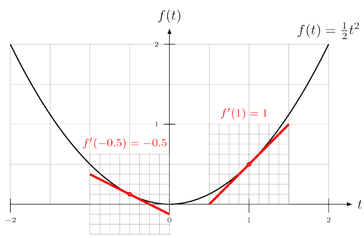
Calculus is the study of functions and their properties. The two operations in the study of calculus are derivatives—which describe how quantities *change over time*—and integrals, which are used to calculate the total amount of a quantity *accumulated* over a time period.

Derivatives

The derivative function $f'(t)$ describes how the function $f(t)$ changes over time. The derivative encodes the information about the *instantaneous rate of change* of the function $f(t)$, which is the same as the *slope* of the graph of the function at that point:

$$f'(t) \equiv \text{slope}_f(t) = \frac{\text{change in } f(t)}{\text{change in } t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

If the derivative $f'(t)$ is equal to 5 units per second, this means that $f(t)$ changes by 5 units each second. The derivative of the constant function is zero because it has zero rise-over-run everywhere. The derivative of the function $f(t) = mt + b$ (a line) is the constant function $f'(t) = m$. More generally, the instantaneous slope of a function is different for different values of t, as illustrated in the figure.

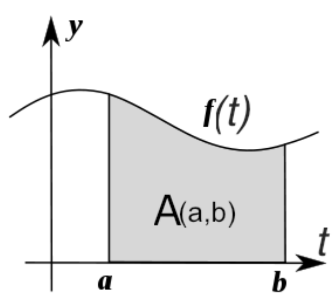


The derivative operation is denoted by several names and symbols: $Df(t) = f'(t) = \frac{df}{dt} = \frac{d}{dt}\{f(t)\} = \dot{f}$ and all these symbols carry the same meaning. Think of $f'(t)$ not as a separate entity from $f(t)$, but as a *property* of the function $f(t)$. It's best to think of the derivative as an *operator* $\frac{d}{dt}$ that you can apply to any function to obtain its slope information.

Integrals

An integral corresponds to the computation of the *area* enclosed between the curve $f(t)$ and the x-axis over some interval:

$$A(a,b) \equiv \int_{t=a}^{t=b} f(t)\, dt.$$

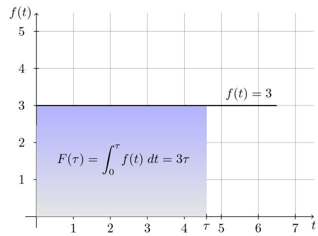


The symbol $\int$ is shorthand for *sum*. Indeed, the area under the curve corresponds to the sum of the values of the function $f(t)$ between $t = a$ and $t = b$.

The integral is the total of f between a and b.

Example 1

We can easily find the area under the graph of the constant function $f(t) = 3$ between any two points because the region under the curve is rectangular. Choosing $t = 0$ as the starting point, we obtain the integral function $F(\tau)$, which corresponds to the area under $f(t)$ between $t = 0$ and $t = \tau$:



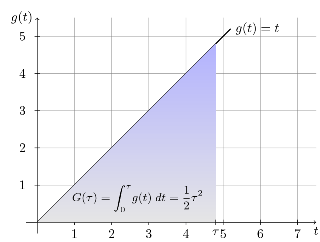
$$F(\tau) \equiv A(0,\tau) = \int_0^\tau f(t)\,dt = 3\tau.$$

The area is equal to the rectangle's height times its width.

Example 2

Consider now the area under the graph of the line $g(t) = t$, starting from $t = 0$. Since the region under the curve is triangular, we can compute its area. Recall the area of a triangle is given by the length of its base times its height divided by 2.

The general formula for the area under $g(t)$ from $t = 0$ until $t = \tau$ is described by the following integral calculation:



$$G(\tau) \equiv A(0,\tau) = \int_0^\tau g(t)\,dt = \frac{\tau \times \tau}{2} = \tfrac{1}{2}\tau^2.$$

We're able to compute the above integrals thanks to the simple geometries of the areas under the graphs. Later in this book (Chapter 5), we'll develop techniques for finding integrals (areas under the curve) of more complicated functions. In fact, there is an entire course called integral calculus, which is dedicated to the task of finding integrals.

But don't worry, you don't need to know *everything* about integrals to learn physics. What is important right now is that you understand the concept of integration. The integral of a function is the area under the graph of the function, which is in some sense the total amount of the function accumulated during some interval of time. For the most part of the first-year physics, the only integral formulas you'll need to know are

$$\int_0^\tau a\,dt = a\tau \qquad \text{and} \qquad \int_0^\tau at\,dt = \tfrac{1}{2}a\tau^2.$$

The first integral describes the general calculation of the area under a constant function, like in Example 1. The second formula is a generalization of the formula we derived in Example 2. Using these formulas in combination, you can compute the integral of an arbitrary line $h(t) = mt + b$ as follows:

$$H(\tau) = \int_0^\tau h(t)\,dt = \int_0^\tau (mt+b)\,dt = \int_0^\tau mt\,dt \;+ \int_0^\tau b\,dt = \tfrac{1}{2}m\tau^2 + b\tau.$$

Regroup

At this point you might be on the fence about the new calculus concepts. On one hand, calculating slopes (derivatives) and areas under the curve (integrals) seem like trivial tasks. On the other hand, seeing five different notations for the derivative and the weird integral sign has probably put some fear in you. You might be wondering whether you *really* need to learn about derivatives and integrals. How often do you have to compute the area under the graph of a function in the real world? It turns out that "calculating the area under a curve" is very useful since it is the "undo operation" for the derivative.

Inverse operations

The integral is the inverse operation of the derivative. Many equations in math and physics involve the *derivative* of some unknown function. Understanding the inverse relationship between integrals and derivatives will allow you to solve for the unknown function in these equations.

You should already be familiar with the inverse relationship between functions. When solving equations (page 4), we use inverse functions to *undo* functions that stand in our way as we try to isolate the unknown x. Similarly, we use the integral operation to *undo* the effects of the derivative operation when we try to solve for some unknown function $f(t)$. For example, suppose $g(t)$ is a known function and we're trying to solve for $f(t)$ in the equation

$$\frac{d}{dt}\{f(t)\} = g(t).$$

Taking the integral on the left-hand side of the equation will undo the derivative operation. To keep the equality true, we must apply the "integrate over t" operation on both sides of the equation. We obtain

$$\int \frac{d}{dt}\{f(t)\}\,dt = \int g(t)\,dt,$$

$$f(t) = \int g(t)\,dt.$$

The take-home message is that every time you want to *undo* a derivative, you can apply the integral operation, however, there is a little technical detail we must clarify to make this statement precise.

The integral isn't *exactly* the inverse of the derivative—there is a tricky extra constant factor that appears when we integrate. Let's analyze in more detail what happens when we perform the combo of the derivative operation followed by the integral operation on some function $f(t)$. Suppose we are given the derivative function $f'(t)$ and asked to integrate it between $t = 0$ and $t = \tau$. Intuitively, this integral corresponds to calculating the **total of the changes** in $f(t)$ during that time interval. Recall the notation for "change in f" $\Delta f \equiv f(\tau) - f(0)$, which we used previously. This notation makes it easy to see how the integral over $f'(t)$ corresponds to the total change in $f(t)$ between $t = 0$ and $t = \tau$:

$$\int_0^\tau f'(t)\, dt \;=\; \Delta f \equiv f(\tau) - f(0).$$

Calculating the total of the instantaneous changes in f during the interval $[0, \tau]$ is the same as finding the change in $f(t)$ between the endpoints of the interval. If we rewrite the above equation to isolate $f(\tau)$, we obtain

$$f(\tau) \;=\; f(0) + \int_0^\tau f'(t)\, dt.$$

Note that the expression for $f(\tau)$ depends on the value of $f(t)$ at $t = 0$, which we call the *initial value* of the function. In physics problems, the initial values of the equations of motion $x(0) \equiv x_i$ and $v(0) \equiv v_i$ are called the *initial conditions*.

Banking example To illustrate how derivative and integral operations apply to the real world, I'll draw an analogy from a scenario that every student is familiar with. Consider the function $\text{ba}(t)$, which represents your bank account balance at time t. Also consider the function $\text{tr}(t)$, which corresponds to the transactions (deposits and withdrawals) on your account.

The function $\text{tr}(t)$ is the derivative of the function $\text{ba}(t)$. If you ask, "how does my balance change over time?" the answer is the function $\text{tr}(t)$. Using mathematical symbols, we can represent this relationship as

$$\text{tr}(t) = \frac{d}{dt}\left\{\text{ba}(t)\right\}.$$

If the derivative is positive, your account balance is growing. If the derivative is negative, your account balance is depleting.

Suppose you have a record of all the transactions on your account $\text{tr}(t)$, and you want to compute the final account balance at the end of

the month. Since $\mathrm{tr}(t)$ is the derivative of $\mathrm{ba}(t)$, you can use an integral (the inverse operation of the derivative) to obtain $\mathrm{ba}(t)$. Knowing the balance of your account at the beginning of the month, you can predict the balance at the end of the month by using the following integral calculation:

$$\mathrm{ba}(30) = \mathrm{ba}(0) + \int_0^{30} \mathrm{tr}(t)\, dt.$$

This calculation makes sense since $\mathrm{tr}(t)$ represents the instantaneous changes in $\mathrm{ba}(t)$. If you want to find the overall change in the account balance from day 0 until day 30, you can compute the total of all the transactions on the account.

We use integrals every time we need to calculate the total of some quantity over a time period. In the next section, we'll see how these integration techniques can be applied to the subject of kinematics, and how the equations of motion for UAM are derived from first principles.

2.4 Kinematics with calculus

To carry out kinematics calculations, all we need to do is plug the initial conditions (x_i and v_i) into the correct equation of motion. But how did Newton come up with the equations of motion in the first place? Now that you know Newton's mathematical techniques (calculus), you can learn to derive the equations of motion by yourself.

Concepts

Recall the kinematics concepts related to the motion of objects:

- t: time
- $x(t)$: position as a function of time
- $v(t)$: velocity as a function of time
- $a(t)$: acceleration as a function of time
- $x_i = x(0), v_i = v(0)$: the initial conditions

Position, velocity, and acceleration revisited

The equations of kinematics are used to predict the motion of objects. Suppose you know the acceleration of the object $a(t)$ at all times t. Can you find $x(t)$ starting from $a(t)$?

The equations of motion $x(t)$, $v(t)$, and $a(t)$ are related:

$$a(t) \xleftarrow{\frac{d}{dt}} v(t) \xleftarrow{\frac{d}{dt}} x(t).$$

The velocity function is the derivative of the position function and the acceleration function is the derivative of the velocity function.

General procedure

If you know the acceleration of an object as a function of time $a(t)$, and you know its initial velocity $v_i = v(0)$, you can find its velocity function $v(t)$ for all later times using integration. This is because the acceleration function $a(t)$ describes the change in the object's velocity. If you know the object started with an initial velocity of $v_i \equiv v(0)$, the velocity at a later time $t = \tau$ is equal to v_i plus the total acceleration of the object between $t = 0$ and $t = \tau$:

$$v(\tau) = v_i + \int_0^{\tau} a(t)\, dt.$$

If you know the initial position x_i and the velocity function $v(t)$, you can find the position function $x(t)$ by using integration. We find the position at time $t = \tau$ by adding all the velocities (the changes in the object's position) that occurred between $t = 0$ and $t = \tau$:

$$x(\tau) = x_i + \int_0^{\tau} v(t)\, dt.$$

The procedure for finding $x(t)$ starting from $a(t)$ can be summarized as follows:

$$a(t) \quad \overset{v_i + \int dt}{\longrightarrow} \quad v(t) \quad \overset{x_i + \int dt}{\longrightarrow} \quad x(t).$$

Next, I'll illustrate how you can apply this procedure to the important special case of an object undergoing uniformly accelerated motion.

Derivation of the UAM equations of motion

Consider an object undergoing uniformly accelerated motion (UAM) with acceleration function $a(t) = a$. Suppose we know the initial velocity of $v_i \equiv v(0)$, and we want to find the velocity at a later time $t = \tau$. We compute $v(\tau)$ using the following integral:

$$v(\tau) = v_i + \int_0^{\tau} a(t)\, dt = v_i + \int_0^{\tau} a\ dt = v_i + a\tau.$$

Velocity as a function of time is given by the initial velocity v_i added to the integral of the acceleration. The integration can be visualized as the calculation of the area of a rectangle, similar to the calculation we saw in Example 1 on page 118.

You can also use integration to find the position function $x(t)$ if you know the initial position x_i and the velocity function $v(t)$. The formula is

$$x(\tau) = x_i + \int_0^\tau v(t)\, dt = x_i + \int_0^\tau (v_i + at)\, dt = x_i + v_i\tau + \tfrac{1}{2}a\tau^2.$$

The integration step can be visualized as the calculation of the area of a triangle with slope a stacked on top of a rectangle of height v_i.

Note that the above calculations required knowing the initial conditions x_i and v_i. These initial values were required because the integral calculations we performed only told us the *change* in the quantities relative to their initial values.

The fourth equation

We can derive the fourth equation of motion,

$$v_f^2 = v_i^2 + 2a(x_f - x_i),$$

by combining the equations of motion $v(t)$ and $x(t)$. Let's see how. Start by squaring both sides of the velocity equation $v_f = v_i + at$ to obtain

$$v_f^2 = (v_i + at)^2 = v_i^2 + 2av_it + a^2t^2 = v_i^2 + 2a[v_it + \tfrac{1}{2}at^2].$$

The term in the square bracket is equal to $\Delta x = x(t) - x_i = x_f - x_i$.

Applications of derivatives

Recall that the velocity and the acceleration functions are obtained by taking derivatives of the position function:

$$x(t) \xrightarrow{\frac{d}{dt}} v(t) \xrightarrow{\frac{d}{dt}} a(t).$$

We just saw how to use integration to follow this chain of operations in reverse to obtain $x(t)$ for the special case of constant acceleration:

$$\begin{aligned}
a(t) &\equiv a,\\
v(t) &\equiv v_i + \int_0^t a(\tau)\, d\tau = v_i + at,\\
x(t) &\equiv x_i + \int_0^t v(\tau)\, d\tau = x_i + v_it + \tfrac{1}{2}at^2.
\end{aligned}$$

Note that, in addition to the integral calculations, the formulas for $v(t)$ and $x(t)$ require some additional information—the initial value of the function.

Earlier we defined the derivative operator $\frac{d}{dt}$ that computes the derivative function $f'(t)$, which tells us the slope of the function $f(t)$. There are several derivative formulas that you need to learn to be proficient at calculus. We'll get to them in Chapter 5. For now, the only derivative formula you'll need is the *power rule* for derivatives:

$$\text{if} \quad f(t) = At^n \quad \text{then} \quad f'(t) = nAt^{n-1}.$$

Using this formula on each term in the function $f(t) = A + Bt + Ct^2$ we find its derivative is $\frac{df}{dt} \equiv f'(t) = 0 + B + 2Ct$.

Let's now use the derivative to verify that the equations of motion we obtained above satisfy $x'(t) = v(t)$ and $v'(t) = a(t)$. Applying the derivative operation to both sides of the equations we obtain

$$\begin{aligned} a'(t) &\equiv 0, \\ v'(t) &\equiv \tfrac{d}{dt}\{v_i + at\} = \cancel{\tfrac{d}{dt}\{v_i\}} + \tfrac{d}{dt}\{at\} = 0 + a &= a(t), \\ x'(t) &\equiv \cancel{\tfrac{d}{dt}\{x_i\}} + \tfrac{d}{dt}\{v_i t\} + \tfrac{d}{dt}\{\tfrac{1}{2}at^2\} = 0 + v_i + at &= v(t). \end{aligned}$$

Note that computing the derivative of a function kills the information about its initial value; the derivative contains only information about the changes in $f(t)$.

Let's summarize what we've learned so far about derivatives and integrals. Integrals are useful because they allow us to compute $v(t)$ from $a(t)$, and $x(t)$ from $v(t)$. The derivative operation is useful because it allows us to obtain $v(t)$ if we know $x(t)$, and/or obtain $a(t)$ if we know $v(t)$. Recall that $x(t)$, $v(t)$, and $a(t)$ correspond to three different aspects of the same motion, as shown in Figure 2.2 on page 112. The operations of calculus allow us to move freely between the different descriptions of the motion.

Discussion

According to Newton's second law of motion, forces are the cause of acceleration and the formula that governs this relationship is

$$F_{\text{net}} = ma,$$

where F_{net} is the magnitude of the net force acting on the object.

In Chapter 4 we'll learn about *dynamics*, the study of the different kinds of forces that can act on objects: gravitational force $\vec{F}_g$, spring force $\vec{F}_s$, friction force $\vec{F}_f$, and other forces. To find an object's acceleration, we must add together all the forces acting on the object and divide by the object's mass:

$$\sum F_i = F_{\text{net}} \qquad \Rightarrow \qquad a = \frac{1}{m} F_{\text{net}}.$$

The physics procedure for predicting the motion of an object given the forces acting on it can be summarized as follows:

$$\frac{1}{m}\underbrace{\left(\sum \vec{F} = \vec{F}_{\text{net}}\right)}_{\text{dynamics}} = \underbrace{a(t) \overset{v_i+\int dt}{\longrightarrow} v(t) \overset{x_i+\int dt}{\longrightarrow} x(t)}_{\text{kinematics}}.$$

Free fall revisited

The force of gravity acting on an object of mass m on the surface of the Earth is given by $\vec{F}_g = -mg\hat{y}$, where $g = 9.81[\text{m/s}^2]$ is the *gravitational acceleration* on the surface of the Earth. We previously discussed that an object is in *free fall* when the only force acting on it is the force of gravity. In this case, Newton's second law tells us

$$\begin{aligned}\vec{F}_{\text{net}} &= m\vec{a}\\ -mg\hat{y} &= m\vec{a}.\end{aligned}$$

Dividing both sides by the mass, we see the acceleration of an object in free fall is $\vec{a} = -9.81\hat{y}$.

It's interesting to note that an object's mass does not affect its acceleration during free fall. The force of gravity is proportional to the mass of the object, but acceleration is inversely proportional to the mass of the object; overall, it holds that $a_y = -g$ for objects in free fall, regardless of their mass. This observation was first made by Galileo in his famous Leaning Tower of Pisa experiment. Galileo dropped a wooden ball and a metal ball (same shape, different mass) from the Leaning Tower of Pisa, and observed that they fell to the ground at the same time. Search for "Apollo 15 feather and hammer drop" on `YouTube` to see this experiment performed on the Moon.

What next?

You might have noticed that in the last couple of paragraphs we started putting little arrows on top of certain quantities. The arrows are there to remind you that forces, velocities, and accelerations are *vector quantities*. In the next chapter, we'll make a short mathematical digression to introduce all the vectors concepts necessary to understand physics.

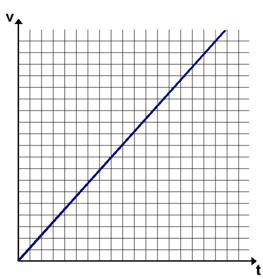
Before we proceed with the vectors lessons and more advanced physics topics, it's a good idea to practice using the physics equations you just learned. Take the time to solve a couple of the practice problems in the next section.

2.5 Kinematics problems

We spent an entire chapter learning about position, velocity, and acceleration equations used to describe the motion of objects. It's now time to practice using these equations to solve problems.

Here are some general tips for solving kinematics problems. First, try to determine which equation you'll need to solve the problem. There are just four of them: $x(t) = x_i + v_i t + \frac{1}{2}at^2$, $v(t) = v_i + at$, $a(t) = a = \frac{F_{\text{net}}}{m}$, and $v_f^2 = v_i^2 + 2a\Delta x$, so it can't be that hard. If you can't figure it out, check the hint then try solving the problem. Always draw a diagram labelling all the variables that appear in your equations. This way you'll always have a picture of what is going on. Check your answer against the answer provided on page 382. If you didn't get the right answer, check your work and try again. Don't look at the solution yet. Try to figure out the problem by revisiting the assumptions you made, the equations you wrote, and the steps you followed. Look at the solution only if you can't figure out the problem after 10 minutes and you're running out of ideas.

P2.1 Below is a velocity-vs-time graph of a moving particle. Is the particle gaining or losing speed? Does the graph describe uniformly accelerated motion (UAM) or not?



Hint: Acceleration is the slope of the velocity graph.

P2.2 You're running away from point A. At $t = 2$[s] you're 3[m] away from A, at $t = 4$[s] you're 8[m] away from A, and at $t = 6$[s] you're 14[m] away from A. Are you running with uniform velocity (UVM)?
Hint: Calculate the velocity during each time interval.

P2.3 A car is moving on a straight road. Indicate whether the car's speed is increasing or decreasing in the following cases:

1. Velocity is negative, acceleration is positive.
2. Velocity is negative, acceleration is negative.

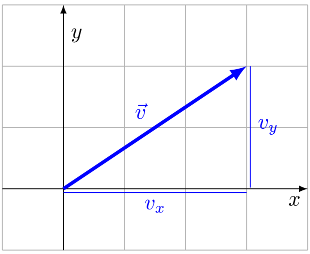
Hint: Pay attention to the relative direction of acceleration to velocity.

Chapter 3

Vectors

In this chapter we'll learn how to manipulate multi-dimensional objects called vectors. Vectors are the precise way to describe directions in space. We need vectors in order to describe physical quantities like the velocity of an object, its acceleration, and the net force acting on the object.

Vectors are built from ordinary numbers, which form the *components* of the vector. You can think of a vector as a list of numbers, and *vector algebra* as operations performed on the numbers in the list. Vectors can also be manipulated as geometrical objects, represented by arrows in space. The arrow that corresponds to the vector $\vec{v} = (v_x, v_y)$ starts at the origin $(0, 0)$ and ends at the point (v_x, v_y). The word vector comes from the Latin *vehere*, which means *to carry*. Indeed, the vector $\vec{v}$ takes the point $(0, 0)$ and carries it to the point (v_x, v_y).



This chapter will introduce you to vectors, vector algebra, and vector operations, which are very useful for solving physics problems. What you'll learn here applies more broadly to problems in computer graphics, probability theory, machine learning, and other fields of science and mathematics. It's all about vectors these days, so you better get to know them.

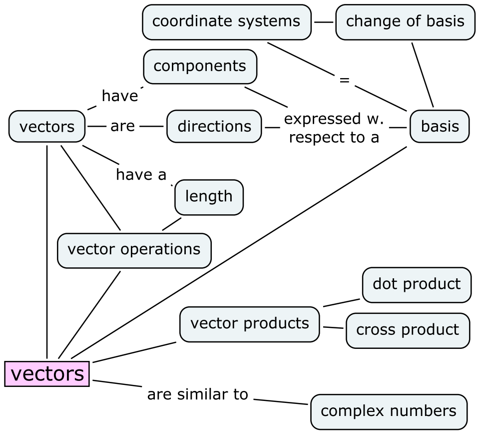


Figure 3.1: This figure illustrates the new concepts related to vectors. As you can see, there is quite a bit of new vocabulary to learn, but don't be fazed—all these terms are just fancy ways of talking about arrows.

3.1 Great outdoors

Vectors are directions for getting from point A to point B. Directions can be given in terms of street names and visual landmarks, or with respect to a coordinate system.

While on vacation in British Columbia, you want to visit a certain outdoor location your friend told you about. Your friend isn't available to take you there himself, but he has sent you *directions* for how to get to the place from the bus stop:

```
Sup G. Go to bus stop number 345. Bring a compass.
Walk 2 km north then 3 km east. You will find X there.
```

This text message contains all the information you need to find X.

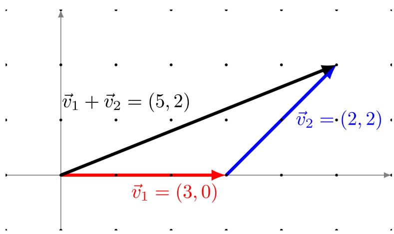
Act 1: Following directions

You arrive at the bus station, located at the top of a hill. From this height you can see the whole valley, and along the hillside below spreads a beautiful field of tall crops. The crops are so tall they prevent anyone standing in them from seeing too far; good thing you have a compass. You align the compass needle so the red arrow points north. You walk 2 km north, then turn right (east) and walk another 3 km. You arrive at X.

Vector as arrows

So far, we described how to perform algebraic operations on vectors in terms of their components. Vector operations can also be interpreted geometrically, as operations on two-dimensional arrows in the Cartesian plane.

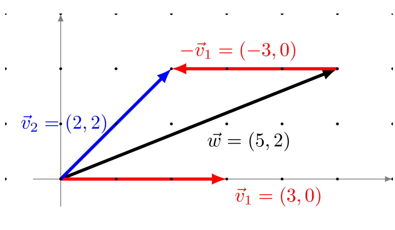
Vector addition The sum of two vectors corresponds to the combined displacement of the two vectors. The diagram on the right illustrates the addition of two vectors, $\vec{v}_1 = (3,0)$ and $\vec{v}_2 = (2,2)$. The sum of the two vectors is the vector $\vec{v}_1 + \vec{v}_2 = (3,0) + (2,2) = (5,2)$.



Vector subtraction Before we describe vector subtraction, note that multiplying a vector by a scaling factor $\alpha = -1$ gives a vector of the same length as the original, but pointing in the opposite direction.

This fact is useful if you want to subtract two vectors using the graphical approach. Subtracting a vector is the same as adding the negative of the vector:

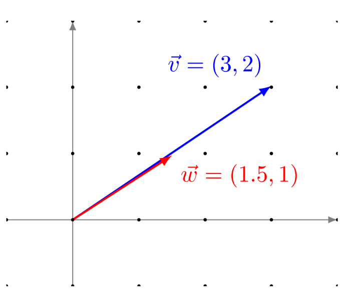
$$\vec{w} - \vec{v}_1 = \vec{w} + (-\vec{v}_1) = \vec{v}_2.$$



The diagram on the right illustrates the graphical procedure for subtracting the vector $\vec{v}_1 = (3,0)$ from the vector $\vec{w} = (5,2)$. Subtraction of $\vec{v}_1 = (3,0)$ is the same as addition of $-\vec{v}_1 = (-3,0)$.

Scaling The scaling operation acts to change the length of a vector. Suppose we want to obtain a vector in the same direction as the vector $\vec{v} = (3,2)$, but half as long. "Half as long" corresponds to a scaling factor of $\alpha = 0.5$. The scaled-down vector is $\vec{w} = 0.5\vec{v} = (1.5,1)$.

Conversely, we can think of the vector $\vec{v}$ as being twice as long as the vector $\vec{w}$.



3.6 Vectors problems

You learned a bunch of vector formulas and you saw some vector diagrams, but did you really learn how to solve problems with vectors? There is only one way to find out: test yourself by solving problems.

I've said it before and I don't want to repeat myself too much, but it's worth saying again: the more problems you solve, the better you'll understand the material. It's now time for you to try the following vector problems to make sure you're on top of things.

P3.1 Express the following vectors in length-and-direction notation:

(a) $\vec{u}_1 = (0, 5)$ (b) $\vec{u}_2 = (1, 2)$ (c) $\vec{u}_3 = (-1, -2)$

P3.2 Express the following vectors as components:

(a) $\vec{v}_1 = 20\angle 30°$ (b) $\vec{v}_2 = 10\angle{-90}°$ (c) $\vec{v}_3 = 5\angle 150°$

P3.3 Express the following vectors in terms of unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$:

(a) $\vec{w}_1 = 10\angle 25°$ (b) $\vec{w}_2 = 7\angle{-90}°$ (c) $\vec{w}_3 = (3, -2, 3)$

P3.4 Given the vectors $\vec{v}_1 = (1, 1)$, $\vec{v}_2 = (2, 3)$, and $\vec{v}_3 = 5\angle 30°$, calculate the following expressions:

(a) $\vec{v}_1 + \vec{v}_2$ (b) $\vec{v}_2 - 2\vec{v}_1$ (c) $\vec{v}_1 + \vec{v}_2 + \vec{v}_3$

P3.5 Starting from the point $P = (2, 6)$, the three displacement vectors shown in Figure 3.2 are applied to obtain the point Q. What are the coordinates of the point Q?

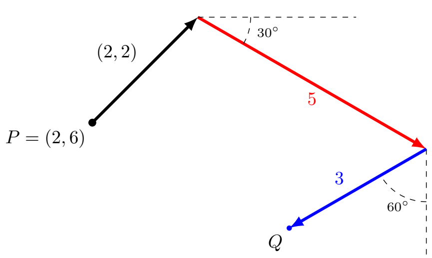


Figure 3.2: A point P is displaced by three vectors to obtain point Q.

P3.6 Given the vectors $\vec{u} = (1, 1, 1)$, $\vec{v} = (2, 3, 1)$, and $\vec{w} = (-1, -1, 2)$, compute the following products:

(1) $\vec{u} \cdot \vec{v}$ (2) $\vec{u} \cdot \vec{w}$ (3) $\vec{v} \cdot \vec{w}$

(4) $\vec{u} \times \vec{v}$ (5) $\vec{u} \times \vec{w}$ (6) $\vec{v} \times \vec{w}$

P3.7 Find a unit-length vector that is perpendicular to both $\vec{u} = (1, 0, 1)$ and $\vec{v} = (1, 2, 0)$.
Hint: Use the cross product.

Chapter 4

Mechanics

4.1 Introduction

Mechanics is the precise study of the motion of objects, the forces acting on them, and more abstract concepts such as momentum and energy. You probably have an intuitive understanding of these concepts already. In this chapter we will learn how to use precise mathematical equations to support your intuition.

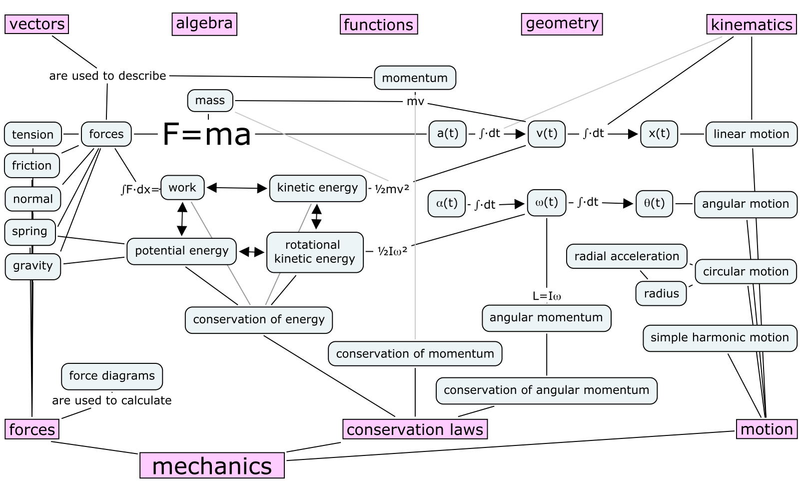


Figure 4.1: The concepts of mechanics. Forces are the cause of motion. We can also analyze the motion of objects in terms of the concepts of energy and momentum. If you understand the connections between all of the above concepts, you understand mechanics.

The above set of equations are similar to the equations we obtained for the fridge that was not moving. The only difference is the kinetic coefficient of friction μ_k replaces the static coefficient of friction μ_s. Keeping the fridge moving with a constant velocity requires an external force $F_{\text{ext}} = \mu_k mg$. Generally, $\mu_k < \mu_s$, so less force is needed to keep the fridge moving than is needed to start the fridge moving.

Let's approach this whole friction thing from a different slant.

Incline

At this point, my dear readers, we're delving into the crucial question that you will—without a doubt—be asked to solve in your homework or at the final exam.

A block is sliding down an incline. What is its acceleration?

Step 1: Draw a diagram that includes the block's weight $\vec{W}$, the normal force $\vec{N}$, and the friction force $\vec{F}_{fk}$.

Step 2: Choose the coordinate system to be tilted along the incline. This is important because, in this coordinate system, the block's motion is purely in the x-direction, while the y-direction remains static.

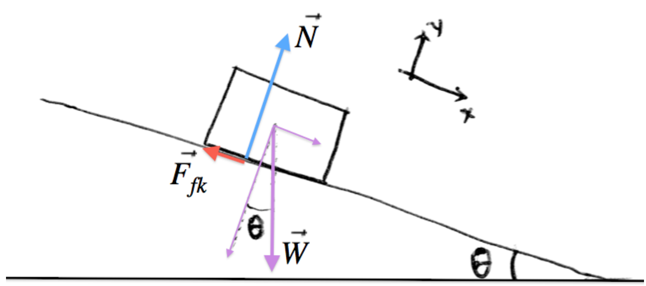


Figure 4.6: A block sliding down an incline with angle θ. What is the block's acceleration?

Steps 3 and 4: Let's copy the empty template and fill in the equations:

$$\sum F_x = \|\vec{W}\| \sin\theta - F_{fk} = ma_x,$$
$$\sum F_y = N - \|\vec{W}\| \cos\theta \quad = 0;$$

or, substituting the values that we know,

$$\sum F_x = mg\sin\theta - \mu_k N = ma_x,$$
$$\sum F_y = N - mg\cos\theta \quad = 0.$$

Step 5: From the y equation, we obtain $N = mg\cos\theta$, which we substitute into the x equation to obtain

$$a_x = \frac{1}{m}(mg\sin\theta - \mu_k mg\cos\theta) = g\sin\theta - \mu_k g\cos\theta.$$

Bathroom scale

You have a spring in your bathroom scale with spring constant k, on which you place a block of mass m. By what length Δy will the spring be compressed?

Step 1, 2: Draw a before and after picture with the y-axis placed at the *natural* length of the spring.

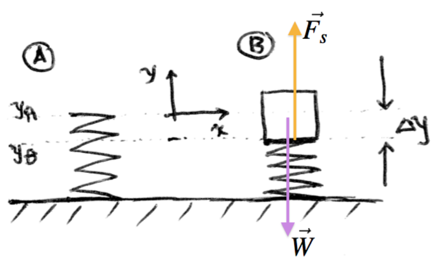


Figure 4.7: A bathroom scale is compressed by a distance Δy when an object of mass m is placed on it.

Steps 3 and 4: Filling in the template, we find

$$\sum F_x = 0 = 0,$$
$$\sum F_y = F_s - mg = 0.$$

Step 5: We know the force exerted by a spring is proportional to its displacement according to

$$F_s = -ky_B,$$

so we can find $y_B = -\frac{mg}{k}$. The length of compression is therefore

$$|\Delta y| = \frac{mg}{k}.$$

Two blocks

Now you're ready for a more involved example with two blocks. One block is sitting on a surface, and the other block is falling straight down. The two blocks are connected by a rope. What is the acceleration of the *system* as a whole?

4.5 Momentum

A collision between two objects creates a sudden spike in the contact force between them, which can be difficult to measure and quantify. It is not possible to use Newton's law $F = ma$ to predict the accelerations that occur during collisions. To predict the motion of the objects after the collision, we need a *momentum* calculation. According to the law of conservation of momentum, the total amount of momentum before and after the collision is the same. Once we know the momenta of the objects before the collision, it becomes possible to calculate their momenta after the collision, and from this determine their subsequent motion.

To illustrate the importance of momentum, consider the following situation. Say you have a 1[g] paper ball and a 1000[kg] car moving at the same speed of 100[km/h]. Which of the two objects would you rather be hit by? Momentum, denoted $\vec{p}$, is the precise physical concept that measures the *quantity* of motion. An object of mass m moving with velocity $\vec{v}$ has a momentum of $\vec{p} \equiv m\vec{v}$. Momentum plays a key role in collisions. Your gut feeling about the piece of paper and the car is correct. The car weighs $1000 \times 1000 = 10^6$ times more than the piece of paper, so the car has 10^6 times more momentum when moving at the same speed. Colliding with the car will "hurt" one-million times more than colliding with the piece of paper, even though both objects approach at the same velocity.

In this section, we'll learn how to use the law of conservation of momentum to predict the outcomes of collisions.

Concepts

- m: the *mass* of the moving object
- $\vec{v}$: the *velocity* of the moving object
- $\vec{p} = m\vec{v}$: the *momentum* of the moving object
- $\sum \vec{p}_{\text{in}}$: the sum of the momenta of particles before a collision
- $\sum \vec{p}_{\text{out}}$: the sum of the momenta of particles after a collision

Definition

The momentum of a moving object is equal to the velocity of the object multiplied by its mass:

$$\vec{p} = m\vec{v} \qquad [\text{kg m/s}]. \tag{4.8}$$

If an object's velocity is $\vec{v} = 20\hat{\imath} = (20, 0)$[m/s] and its mass is 100[kg], then its momentum is $\vec{p} = 2000\hat{\imath} = (2000, 0)$[kg m/s].

Momentum is a vector quantity, and we will often need to convert momentum from the length-and-direction form into the component form:

$$\vec{p} = \|\vec{p}\|\angle\theta = (\|\vec{p}\|\cos\theta, \|\vec{p}\|\sin\theta) = (p_x, p_y).$$

The component form makes it easy to add and subtract vectors: $\vec{p}_1 + \vec{p}_2 = (p_{1x} + p_{2x}, p_{1y} + p_{2y})$. To express the final answer, we will need to convert the component form back to the length-and-direction form:

$$\|\vec{p}\| = \sqrt{p_x^2 + p_y^2}, \qquad \theta = \tan^{-1}\left(\frac{p_y}{p_x}\right).$$

Conservation of momentum

Newton's first law states that in the absence of acceleration ($\vec{a} = 0$), an object maintains a constant velocity. This becomes kind of obvious if you apply the logic of calculus: $\vec{a}$ is the change in $\vec{v}$, so if $\vec{a} = 0$ then $\vec{v}$ must be constant.

In the absence of acceleration, objects conserve their velocity: $\vec{v}_{\text{in}} = \vec{v}_{\text{out}}$. When we multiply both sides of this equation by the object's mass, we obtain an equivalent statement saying that objects conserve their momentum:

$$\vec{p}_{\text{in}} = m\vec{v}_{\text{in}} \;=\; m\vec{v}_{\text{out}} = \vec{p}_{\text{out}}.$$

More generally, for situations involving multiple moving objects, the *sum* of the momenta of all the objects stays constant even if the objects interact. This reasoning is useful when analyzing collisions, since it allows us to equate the sum of the momenta before and after the collision:

$$\sum \vec{p}_{\text{in}} = \sum \vec{p}_{\text{out}}. \tag{4.9}$$

Any momentum that goes into a collision must also come out. This equation expresses the law of conservation of momentum.

The law of conservation of momentum is one of the furthest-reaching laws of physics you will learn by studying mechanics. We discussed the conservation of momentum in the simple context of two colliding particles, but the law applies widely, to multiple particles, fluids, fields, and even collisions involving atomic particles described by quantum mechanics. The quantity of motion (a.k.a. momentum) cannot be created or destroyed—it can only be exchanged between systems.

Examples

Example 1 It's a rainy day, and from your balcony you throw—horizontally, at a speed of 10[m/s]—a piece of rolled-up carton with a

mass of 0.4[g]. Shortly after it leaves your hand, the piece collides with a rain drop that weighs 2[g] and is falling straight down at a speed of 30[m/s]. What will the resulting velocity be if the two objects stick together after the collision?

The conservation of momentum equation says,

$$\vec{p}_{\text{in},1} + \vec{p}_{\text{in},2} = \vec{p}_{\text{out}}.$$

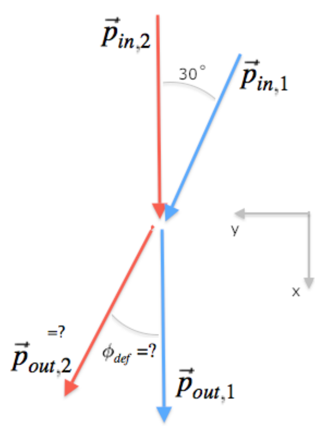
Plugging in the values, we obtain the equation

$$\begin{array}{ccccc} m_1\vec{v}_1 & + & m_2\vec{v}_2 & = & (m_1+m_2)\vec{v}_{\text{out}}, \\ 0.4\times(10,0) & + & 2\times(0,-30) & = & 2.4\times\vec{v}_{\text{out}}. \end{array}$$

Solving for $\vec{v}_{\text{out}}$ we find

$$\vec{v}_{\text{out}} = \frac{0.4(10,0)+2(0,-30)}{2.4} = (1.666,-25.0) = 1.666\hat{\imath} - 25.0\hat{\jmath}.$$

Example 2: Hipsters on bikes Two hipsters on fixed-gear bikes are headed toward the same intersection. Both hipsters have a speed of 50[km/h]. The first hipster crosses the street at a diagonal of 30 degrees when the two bikers collide. Did anyone else see this coming? Apparently, the second hipster didn't, because the thick frames of his glasses were blocking his peripheral vision.



Let's look at the hipster moving in the straight line, and let's assume the combined weight of the hipster and his bike is 100[kg]. As for the street-crossing-at-30-degrees hipster, his weight combined with the weight of his bike frame (which is lighter and more expensive) totals 90[kg].

The story will continue in a moment, but first let's review the information I've given you so far:

$$\begin{aligned}\vec{p}_{\text{in},1} &= 90\times 50\angle 30 \\ &= 90(50\cos 30, 50\sin 30),\end{aligned}$$

$$\begin{aligned}\vec{p}_{\text{in},2} &= 100\times 50\angle 0 \\ &= 100(50,0),\end{aligned}$$

where the x-coordinate points down the street, and the y-coordinate is perpendicular to the street.

Surprisingly, nobody gets hurt in this collision. The bikers bump shoulder-to-shoulder and bounce off each other. The hipster who was trying to cross the street is redirected down the street, while the hipster travelling down the street is deflected to the side and rerouted onto a bike path. I know what you are thinking: couldn't they get hurt at least a little bit? Okay, let's say the whiplash from the shoulder-to-shoulder collision sends the hipsters' heads flying toward each other and smashes their glasses. There you have it.

Suppose the velocity of the first hipster after the collision is 60 [km/h]. What is the velocity and the deflected direction of the second hipster? As given above, the outgoing momentum of the first hipster is $\vec{p}_{\text{out},1} = 90(60, 0)$, and we're looking to find $\vec{p}_{\text{out},2}$.

We can solve this problem with the conservation of momentum formula, which tells us that

$$\vec{p}_{\text{in},1} + \vec{p}_{\text{in},2} \;=\; \vec{p}_{\text{out},1} + \vec{p}_{\text{out},2}.$$

We know three of the above quantities, so we can solve for the remaining unknown vector by isolating it on one side of the equation:

$$\vec{p}_{\text{out},2} = \vec{p}_{\text{in},1} + \vec{p}_{\text{in},2} - \vec{p}_{\text{out},1},$$

$$\vec{p}_{\text{out},2} = 90(50\cos 30, 50\sin 30) + 100(50, 0) - 90(60, 0).$$

The x-component of the momentum $\vec{p}_{\text{out},2}$ is

$$p_{\text{out},2,x} = 90 \times 50\cos 30 + 5000 - 90 \times 60 = 3497.11,$$

and the y-component is $p_{\text{out},2,y} = 90 \times 50\sin 30 = 2250$.

The magnitude of the momentum of hipster 2 is given by

$$\|\vec{p}_{\text{out},2}\| = \sqrt{p^2_{\text{out},2,x} + p^2_{\text{out},2,y}} = 4158.39 \quad [\text{kg km/h}].$$

Note the unit of the momentum is not the standard choice [kg m/s]. That is fine. As long as you keep in mind which units you're using, it's not always necessary to convert to SI units.

The final velocity of hipster 2 is $v_{\text{out},2} = 4158.39/100 = 41.58$[km/h]. The deflection angle is obtained by

$$\phi_{\text{def}} = \tan^{-1}\!\left(\frac{p_{\text{out},2,y}}{p_{\text{out},2,x}}\right) = 32.76°.$$

Discussion

We previously defined the concept of momentum in terms of an object's velocity; but in fact, momentum can be traced to a concept more fundamental than velocity. If you go on to take more advanced

physics classes, you'll learn about the *natural* variables—position and momentum $(\vec{x}, \vec{p})$—that describe the *state* of a particle. You'll also learn that the *real* form of Newton's second law is written in terms of momentum:

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \text{for } m \text{ constant} \;\Rightarrow \quad \vec{F} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}.$$

In most physics problems, objects will maintain a constant mass, so using $\vec{F} = m\vec{a}$ is perfectly fine.

The law of conservation of momentum follows from Newton's third law: for each force $\vec{F}_{12}$ exerted by Object 1 on Object 2, there exists a counter force $\vec{F}_{21}$ of equal magnitude and opposite direction, which is the force of Object 2 pushing back on Object 1. Earlier, I mentioned it is difficult to quantify the magnitude of the exact forces $\vec{F}_{12}$ and $\vec{F}_{21}$ that occur during a collision. Indeed, the amount of force suddenly shoots up as the two objects collide, then suddenly drops again. Complicated as these forces may be, we know that during the entire collision they obey Newton's third law. Assuming there are no other forces acting on the objects, we have

$$\vec{F}_{12} = -\vec{F}_{21} \quad \text{using the above} \Rightarrow \quad \frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}.$$

If we move the negative term to the left-hand side of the equation we obtain

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 = \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right).$$

The second part of the equation implies that the quantity $(\vec{p}_1 + \vec{p}_2)$ is constant over time, and so $\vec{p}_{\text{in},1} + \vec{p}_{\text{in},2} = \vec{p}_{\text{out},1} + \vec{p}_{\text{out},2}$.

In this section, we saw how to use a momentum calculation to predict the motion of particles after a collision. In the next section we'll learn about *energy*, which is another useful concept for understanding and predicting the motion of objects.

Links

[Animations of simple collisions between objects]
`http://en.wikipedia.org/wiki/Conservation_of_linear_momentum`

Exercise

E4.1 A sticky ball of mass 3[g] and velocity 20[m/s] collides with a stationary ball of mass 5[g]. The balls stick together. What is their velocity after the collision?
Hint: Use conservation of momentum $\vec{p}_{1,\text{in}} + \vec{p}_{2,\text{in}} = \vec{p}_{\text{out}}$.

4.6 Energy

Instead of thinking in terms of velocities $v(t)$ and motion trajectories $x(t)$, we can solve physics problems by using *energy* calculations. In this section, we'll precisely define different kinds of energies, and we'll learn the rules for converting one energy into another. The key idea to keep in mind is the principle of *total energy conservation*, which says that in any physical process, the sum of the initial energies is equal to the sum of the final energies.

Example

You drop a ball from a height h[m] and want to predict its speed just before it hits the ground. Through the kinematics approach, you would set up the general equation of motion,

$$v_f^2 = v_i^2 + 2a(y_f - y_i),$$

substitute $y_i = h$, $y_f = 0$, $v_i = 0$, and $a = -g$, and solve for the ball's final speed at impact v_f. The answer is $v_f = \sqrt{2gh}$[m/s].

Alternately, we can use an energy calculation. The ball starts from a height h, which means it has $U_i = mgh$[J] of potential energy. As the ball falls, potential energy is converted into kinetic energy. Just before the ball hits the ground, its final kinetic energy is equal to the initial potential energy: $K_f = U_i$. Since the formula for kinetic energy is $K = \frac{1}{2}mv^2$[J], we have $\frac{1}{2}mv_f^2 = mgh$. We cancel the mass on both sides of the equation and solve for v_f to obtain $v_f = \sqrt{2gh}$[m/s].

Both methods of solving the example problem lead us to the same conclusion, but the energy reasoning is arguably more intuitive than blindly plugging values into a formula. In science, it is really important to know different ways of arriving at the same answer. Knowing about these alternate routes will allow you to check your answers and better understand concepts.

Concepts

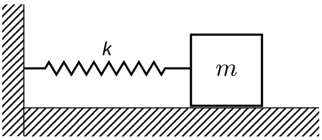
Energy is measured in Joules [J] and it arises in several contexts:

- K = **kinetic energy**: the type of energy objects have by virtue of their motion
- W = **work**: the amount of energy an external force adds or subtracts from a system. Positive work corresponds to energy added to the system while negative work corresponds to energy withdrawn from the system.

4.9 Simple harmonic motion

Vibrations, oscillations, and waves are everywhere around us. For example, what appears to our eyes as white light is actually made of many different oscillations of the electromagnetic field. These oscillations vibrate at a range of frequencies, which correspond to the colours we perceive. Sounds are also made of combined air vibrations with various frequencies and strengths. In this section, we'll learn about *simple harmonic motion*, which describes the oscillation of a mechanical system at a fixed frequency and with a constant amplitude. As its name suggests, simple harmonic motion is the simplest form of oscillatory motion. By studying oscillations in their simplest form, you'll gather important intuition that applies to all types of oscillations and wave phenomena.

The canonical example of simple harmonic motion is the motion of a mass-spring system, illustrated in the figure. The block is free to slide along the horizontal frictionless surface. If the system is disturbed from its equilibrium position, it will start to oscillate back and forth at a certain *natural* frequency, which depends on the mass of the block and the stiffness of the spring.



We'll focus our attention on two mechanical systems: the mass-spring system and the simple pendulum. We'll follow the usual approach by describing the positions, velocities, accelerations, and energies associated with these two kinds of motion. The notion of *simple harmonic motion* (SHM) reaches further than these two systems. The equations and intuition developed while analyzing the oscillations within these simple mechanical systems can be applied more generally to sound oscillations, electric current oscillations, and even quantum oscillations. Pay attention, is all I'm saying.

Concepts

- A: The *amplitude* of the movement is how far the object moves back and forth relative to its centre position.
- $x(t)$[m], $v(t)$[m/s], $a(t)$[m/s^2]: position, velocity, and acceleration of the object as functions of time
- T[s]: the *period* of the object's motion. The period is how long it takes for the motion to repeat.
- f[Hz]: the *frequency* of the motion
- ω[rad/s]: *angular frequency*
- ϕ[rad]: the phase shift denoted by the Greek letter *phee*

4.11 Mechanics problems

It's now time for you to verify experimentally how well you've understood the material from this chapter. Try solving the physics problems presented in this section. Go ahead, dig in! And don't be discouraged if you find some of the problems difficult—they are meant to be challenging in order to force you to think hard and reinforce the connections between the concepts in your head.

When solving physics problems, I recommend you follow this five-step procedure:

1. Figure out what **type of problem** you are dealing with. Is it a kinematics problem? A momentum problem? An energy problem? A problem about angular motion?

2. Draw a **diagram** that describes the physical situation. If the problem involves vectors, draw a coordinate system. Label the known and the unknown quantities in the diagram.

3. Write down the physics **formulas** that are usually used for the type of problem you're solving. You can copy the necessary formulas from the table on page 432.

4. Substitute the **known quantities** into the equations and determine which unknown(s) you need to find. Visualize the steps you'll take to solve for the unknown(s).

5. Do the math.

Note that math appears only in the last step. If you want to solve a physics problem and the first thing you do is manipulate equations and numbers, you're shooting yourself in the foot. Physics is not about solving equations; rather, the focus of physics is thinking abstractly about the "moving parts" in the problem: positions, velocities, energies, etc. As far as I'm concerned, if you complete Steps 1, 2, 3, and 4 correctly and make a mistake in Step 5, you're good in my books. Manipulating math equations fluently and errorlessly is a skill that takes time to hone. If you're still new to the techniques covered in Chapters 1–3, it's normal to make mistakes. Don't worry about it; just practice.

Make sure you attempt each of the exercises on your own before looking at the answers and the solutions. If you want to practice Step 1 of the "solving physics problems" procedure, don't look at the hints. The first step, determining the type of problem, is very strategic and you need to practice it. The problems are intentionally presented out of order, to force you to think about Step 1. Knowing what type of problem you're dealing with is the part that most closely resembles

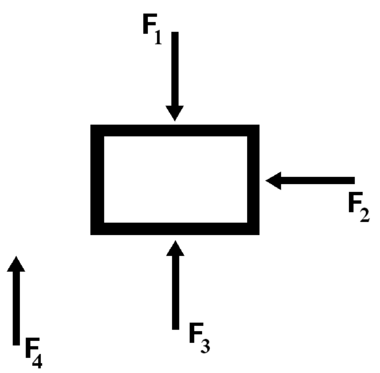
what physics research is like. Given a physics question, physicists try to visualize the situation, label the variables of the problem, and then ask "What can I use here?" Earlier, I likened working with physics equations to playing with LEGOs. You must find the physics equation (or principle) that "fits" the problem. Once you've identified the type of problem, writing the equations and doing the math become comparatively easier tasks. The cool part about learning physics in the "controlled environment" of this problem set is that one of the equations you learned is guaranteed to work.[1]

P4.1 You throw a water balloon from ground level with initial velocity $\vec{v}_i$ at an angle θ above the horizontal.

1. Find $v_y(t)$, the vertical velocity of the balloon as a function of time: (a) when the y-axis points up and (b) when the y-axis points down.
2. A cat starts running away from you just as the you throw the balloon. If the cat's horizontal velocity v_{cat} is equal to v_{ix} of the balloon, will the cat get splashed by the balloon?

Hint: When y is upward, a_y is negative. The balloon's v_x is constant.

P4.2 The four vectors in the diagram below have the same magnitude. Place $\vec{F}_4$ properly (you can change its direction) to achieve the following cases: (1) $\vec{a}_{\text{block}} = 0$, (2) $\vec{a}_{\text{block}}$ has an upward component, and (3) $\vec{a}_{\text{block}}$ has a single component directed to the left.
Hint: Use the equation $\vec{F}_{\text{net}} = m\vec{a}$.

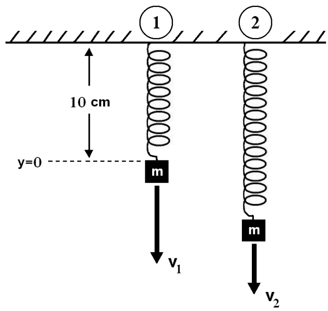


P4.3 Two particles: the first has mass m and speed $2v$, the second has mass $2m$ and speed v. Compare the magnitudes of their momenta and their kinetic energies.
Hint: Recall $\|\vec{p}\| = m\|\vec{v}\|$ and $K = \frac{1}{2}m\|\vec{v}\|^2$.

[1] In research, this isn't always the case; sometimes there is no "known" strategy to follow and a new approach is needed to solve the problem.

P4.4 A space station has two identical compartments A and B and is moving with velocity $\vec{v}$ in space. An explosive charge separates the two compartments and they continue with velocities $\vec{v}_{\mathrm{A}}$ and $\vec{v}_{\mathrm{B}}$. Find $\vec{v}_{\mathrm{B}}$ in the following three cases: (1) $\vec{v}_{\mathrm{A}} = \vec{v}$, (2) $\vec{v}_{\mathrm{A}} = -\vec{v}$, and (3) $\vec{v}_{\mathrm{A}} = 0$.
Hint: Use conservation of momentum.

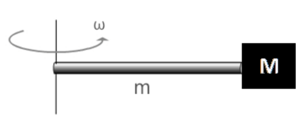
P4.5 A 10[cm] spring is suspended vertically and a mass m hangs from it. What are the types of energies in the system when the mass m is in positions 1 and 2 below? Measure U_g relative to the height $y = 0$.



P4.6 You throw a ball from the ground vertically at speed v and measure its speed when it falls back to the ground. You first carry out this experiment on Earth, then repeat it on the Moon. Does the ball have a greater speed as it hits the ground on Earth or on the Moon?
Hint: Use conservation of energy.

P4.7 In the previous problem, assume there is a pit that allows the ball to fall 10[m] below the level from which it was thrown. Will the ball have a greater speed on Earth or on the Moon when it hits the bottom of the 10[m]-deep pit?
Hint: Use conservation of energy.

P4.8 A rod of mass m is rotating horizontally about one of its ends. An additional mass M is attached at the other end. Assume the system rotates at a constant angular velocity ω. What is the torque on the mass M? If the mass M is detached from the rod without any intervention of an external force, what will the new angular velocity of the rod be?



Hint: Use $\mathcal{T} = I\alpha$. Use conservation of angular momentum.

P4.9 A car is driving inside a vertical circular loop-de-loop. Assume the car passes the top and the bottom of the loop at the same speed. Will the normal force exerted by the loop on the car be greater at the top or at the bottom?
Hint: The car requires a centripetal force to maintain its circular path.

Chapter 5

Calculus

Calculus is *useful* math. We use calculus to solve problems in physics, chemistry, computing, biology, and many other areas of science. You need calculus to perform the quantitative analysis of how functions change over time (derivatives), and to calculate the total amount of a quantity that accumulates over a time period (integrals).

The language of calculus will allow you to speak precisely about the properties of functions and better understand their behaviour. You will learn how to calculate the slopes of functions, how to find their maximum and minimum values, how to compute their integrals, and other tasks of practical importance.

5.1 Introduction

In Chapter 2, we developed an intuitive understanding of integrals. Starting with the knowledge of an object's acceleration function over time, we used the integration operation to calculate the object's velocity function and its position function. We'll now take a closer look at the techniques of calculus using precise mathematical statements, and study how these techniques apply to other problems in science.

A strong knowledge of functions is essential for your understanding of the new calculus concepts. I recommend revisiting Section 1.14 (page 46) to remind yourself of the functions introduced therein. I insist on this. Go! Seriously, there is no point in learning that the derivative of the function $\sin(x)$ is the function $\cos(x)$ if you don't have a clue what $\sin(x)$ and $\cos(x)$ are.

Before we introduce any formal definitions, formulas or derivations, let's demonstrate how calculus is used in a real-world example.

Download example

Suppose you're downloading a large file to your computer. At $t = 0$ you click "save as" in your browser and the download starts. Let $f(t)$ represent the size of the downloaded data. At any time t, the function $f(t)$ tells you the amount of disk space taken by the partially-downloaded file. You are downloading a 720[MB] file, so the download progress at time t corresponds to the fraction $\frac{f(t)}{720[\text{MB}]}$.

Download rate

The derivative function $f'(t)$, pronounced "f prime," describes how the function $f(t)$ changes over time. In our example $f'(t)$ is the download speed. If your downloading speed is $f'(t) = 100[\text{kB/s}]$, then the file size $f(t)$ must increase by 100[kB] each second. If you maintain this download speed, the file size will grow at a constant rate: $f(0) = 0[\text{kB}]$, $f(1) = 100[\text{kB}]$, $f(2) = 200[\text{kB}]$, ..., $f(100) = 10[\text{MB}]$.

To calculate the "estimated time remaining" until the download's completion, we divide the amount of data that remains to be downloaded by the current download speed:

$$\text{time remaining} = \frac{720 - f(t)}{f'(t)} \quad [\text{s}].$$

The bigger the derivative, the faster the download will finish. If your Internet connection were 10 times faster, the download would finish 10 times more quickly.

Inverse problem

Let's consider this situation from the point of view of the modem that connects your computer to the Internet. Any data you download comes through the modem. The modem knows the download rate $f'(t)[\text{kB/s}]$ at all times during the download.

However, since the modem is separate from your computer, it does not know the file size $f(t)$ as the download progresses. Nevertheless, the modem can infer the file size at time t from knowing the transmission rate $f'(t)$. The integral of the download rate between $t = 0$ and $t = \tau$ corresponds to the total amount of downloaded data stored on your computer. During this download period, the change in file size is described by the integral

$$\Delta f = f(\tau) - f(0) = \int_0^\tau f'(t)\, dt\,.$$

Assuming the file size starts from zero $f(0) = 0[\text{kB}]$ at $t = 0$, the modem can use the integration procedure to find $f(\tau)$, the file size on

your computer at $t = \tau$:

$$f(\tau) = \int_0^\tau f'(t)dt\,.$$

The download rate $f'(t)$ is measured in [kB/s], and each time step dt is 1[s] long, so the data downloaded during one second is $f'(t)dt$[kB]. The file size at time $t = \tau$ is equal to the sum of the data downloaded during each second from $t = 0$ until $t = \tau$.

The integral $\int_a^b q(t)\,dt$ is the calculation of the *total* of some quantity $q(t)$ that accumulates during the time period from $t = a$ to $t = b$. Integrals are necessary any time you want to calculate the total of a quantity that changes over time.

As demonstrated above, calculus is much more than the theoretical activity reserved for math specialists. Calculus relates to everyday notions you're already familiar with. Indeed, we carry out calculus-like operations in our head every day—we just don't necessarily use calculus terminology when we do so.

Learning the language of calculus will help you think more clearly about certain types of problems. Understanding the language of calculus is *essential* for learning science because many laws of nature are best described in terms of derivatives and integrals.

Usually, differential calculus and integral calculus are taught as two separate subjects. Perhaps teachers and university administrators are worried the undergraduates' little heads will explode from sudden exposure to *all* of calculus. However, this separation actually makes calculus more difficult, and prevents students from discovering the connections between differential and integral calculus. We'll have no such split in this book, because I believe you can handle the material in one go. Understanding calculus involves figuring out new mathematical concepts like infinity, limits, and summations, but these ideas are not *that* complicated. By getting this far, you've proven you're more than ready to learn the theory, techniques, and applications of derivatives, integrals, sequences, and series.

Let's begin with an overview of the material.

5.2 Overview

This section presents a bird's-eye view of the core concepts of calculus. We'll define more precisely the operations of differentiation and integration, which were introduced in Chapter 2 (see page 116). We'll also discuss the other parts of calculus: *limits*, *sequences*, and *series*. We'll briefly touch upon some applications for each of these concepts; after all, you should know *why* you want to learn all this stuff.

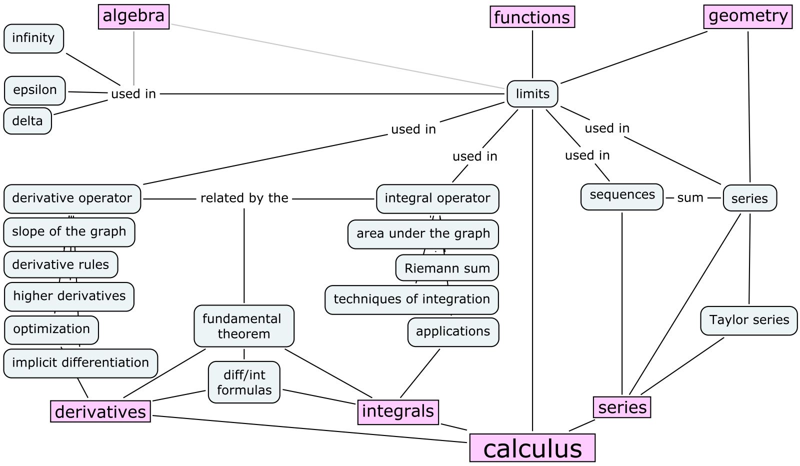


Figure 5.1: The main topics in calculus are limits, derivatives, integrals, sequences, and series. Understanding these notions and how they relate will equip you with many practical problem-solving skills.

Calculus requires a higher level of abstraction than the mathematical topics discussed in Chapter 1. We began our journey through "Math Land" with the study of *numbers*. Then we learned about *functions*, which are transformations that take real numbers as inputs and produce real numbers as outputs, $f : \mathbb{R} \to \mathbb{R}$. In calculus, the derivative and integral *operators* are procedures that take functions as inputs and produce functions as outputs. Let $\{\mathbb{R} \to \mathbb{R}\}$ denote the set of all functions that take real numbers as inputs and produce real numbers as outputs. The derivative operator takes functions as inputs and produces functions as outputs:

$$\frac{d}{dx} : \{\mathbb{R} \to \mathbb{R}\} \quad \to \quad \{\mathbb{R} \to \mathbb{R}\}.$$

More specifically, the derivative operator $\frac{d}{dx}$ acts on a function $f(x)$ to produce its derivative function: $\frac{d}{dx}\left[f(x)\right] = f'(x)$.

Differential calculus

Consider the function $f(x)$, which takes real numbers as inputs and produces real numbers as outputs, $f : \mathbb{R} \to \mathbb{R}$. The input variable for the function f is usually denoted x, but we will sometimes also use the variables u, t, and τ to denote the inputs. The function's output is denoted $f(x)$ and is usually identified with the y-coordinate in graphs.

The *derivative* function, denoted $f'(x)$, $\frac{d}{dx}f(x)$, $\frac{df}{dx}$, or $\frac{dy}{dx}$, describes the *rate of change* of the function $f(x)$. For example, the constant function $f(x) = c$ has derivative $f'(x) = 0$ since the function $f(x)$ does not change at all.

The derivative function describes the *slope* of the graph of the function $f(x)$. The derivative of a line $f(x) = mx + b$ is $f'(x) = m$ since the slope of this line is equal to m. In general, the slope of a function is different at different values of x. For a given choice of input $x = x_0$, the value of the derivative function $f'(x_0)$ is equal to the slope of $f(x)$ as it passes through the point $(x_0, f(x_0))$.

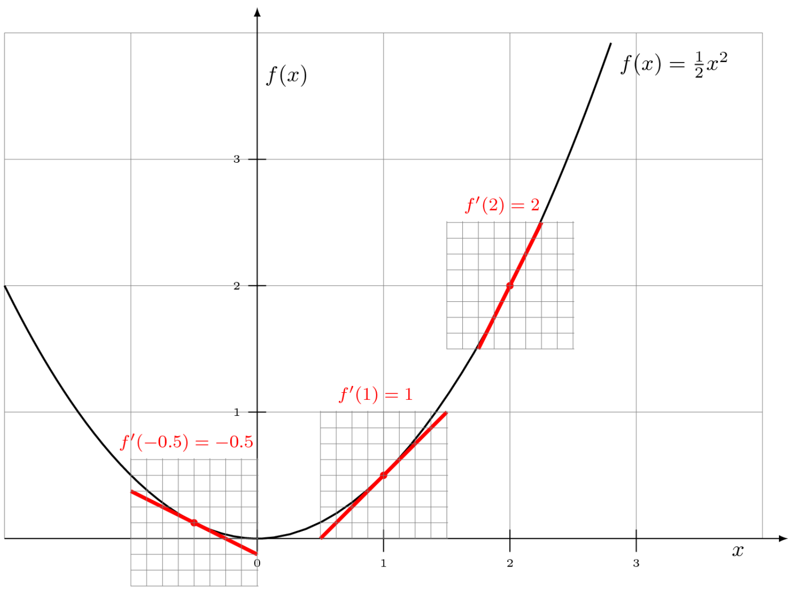


Figure 5.2: The diagram illustrates how to compute the derivative of the function $f(x) = \frac{1}{2}x^2$ at three different points on the graph of the function. To calculate the derivative of $f(x)$ at $x = 1$, we can "zoom in" near the point $(1, \frac{1}{2})$ and draw a line that has the same slope as the function. We can then calculate the slope of the line using a rise-over-run calculation, aided by the mini coordinate system that is provided. The derivative calculations for $x = -\frac{1}{2}$ and $x = 2$ are also shown. Note that the slope of the function is different for each value of x. What is the value of the derivative at $x = 0$? Can you find the general pattern?

The derivative function $f'(x)$ describes the slope of the graph of the function $f(x)$ for all inputs $x \in \mathbb{R}$. The derivative function is a function of the form $f' : \mathbb{R} \to \mathbb{R}$. In our study of mechanics, we learned about the position function $x(t)$ and the velocity function $v(t)$, which describe the motion of an object over time. The velocity is the deriva-

tive of the object's position with respect to time $v(t) = \frac{dx}{dt} = x'(t)$.

The derivative function $f'(x)$ is a property of the original function $f(x)$. Indeed, this is where the name *derivative* comes from: $f'(x)$ is not an independent function—it is *derived* from the original function $f(x)$. In mechanics, the function $x(t)$ describes an object's position as a function of time, and the velocity function $v(t)$ describes one property of the position function, namely, how fast the object's position is changing. Similarly, the acceleration function $a(t)$ describes the rate of change of the function $v(t)$.

The *derivative operator*, denoted $\frac{d}{dx}$ or simply D, takes as input a function $f(x)$ and produces as output the derivative function $f'(x)$. The derivative operator notation is useful because it show the derivative is an operation you do to a function:

$$f'(x) = \frac{d}{dx} f(x).$$

The derivative operator acts on the original function $f(x)$ to produce the derivative function $f'(x)$, which describes the rate of change of f for all x. Applying the derivative operator to a function is also called "taking the derivative" of a function.

For example, the derivative of the function $f(x) = \frac{1}{2}x^2$ is the function $f'(x) = x$. We can describe this relationship as $(\frac{1}{2}x^2)' = x$ or as $\frac{d}{dx}(\frac{1}{2}x^2) = x$. You should flip back to Figure 5.2 and use the graph to prove to yourself that the slope of $f(x) = \frac{1}{2}x^2$ is described by $f'(x) = x$ everywhere on the graph.

Differentiation techniques

Section 5.6 will formally define the derivative operation. Afterward, we'll develop techniques for computing derivatives, or *taking* derivatives. Computing derivatives is not a complicated task once you learn how to use the derivative formulas. If you flip ahead to Section 5.7 (page 270), you'll find a table of formulas for taking the derivatives of common functions. In Section 5.8, we'll learn the basic rules for computing derivatives of sums, products, and compositions of the basic functions.

Applications of derivatives

Once you develop your ability to find derivatives, you'll be able to use this skill to perform several useful tasks.

Optimization The most prominent application of differential calculus is *optimization*: the process of finding a function's maximum and minimum values. When a function reaches its maximum value,

its derivative momentarily becomes zero. The function increases just before it reaches its maximum, and the function decreases just after its maximum. At its maximum value, the function is horizontal, and $f'(x) = 0$ at this point.

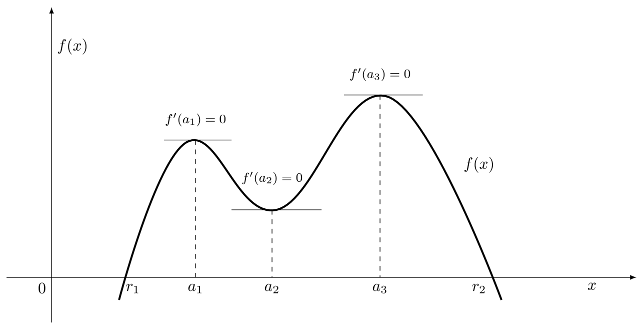


Figure 5.3: The *critical points* of a function occur where the function's derivative equals zero. The critical points of the illustrated function $f(x)$ are $x = a_1$, $x = a_2$, and $x = a_3$. You can use the critical points to find the location of a function's maxima and minima. The point $(a_1, f(a_1))$ is called a *local maximum* of the function, the point at $x = a_2$ is a *local minimum*, while the point at $x = a_3$ is the function's *global maximum*.

The values of x for which $f'(x) = 0$ are called the *critical points* of the function $f(x)$. To find the maximum of a function, we start by compiling a list of its critical points, then go through the list to find the point where the function takes on its largest value. We will discuss the details of this optimization algorithm in Section 5.10.

Tangent lines The *tangent line* to the function $f(x)$ at $x = x_0$ corresponds to the line that passes through the point $(x_0, f(x_0))$ and has the same slope as the function at that point. The word *tangent* comes from the Latin *tangere*, meaning "to touch."

The tangent line to the function $f(x)$ at the point $x = x_0$ is described by the equation

$$T_1(x) = \underbrace{f'(x_0)}_{m} x \ + \ \underbrace{(f(x_0) - f'(x_0)x_0)}_{b} = f(x_0) \ + \ f'(x_0)(x - x_0).$$

The tangent line $T_1(x)$ is an approximation to the function $f(x)$ near the coordinate $x = x_0$. The approximation $T_1(x)$ is equal to the function $f(x)$ at $x = x_0$ since the tangent line passes through the point $(x_0, f(x_0))$. For coordinates near $x = x_0$, the approximation is also accurate since $T_1(x)$ has the same slope as the function $f(x)$. As

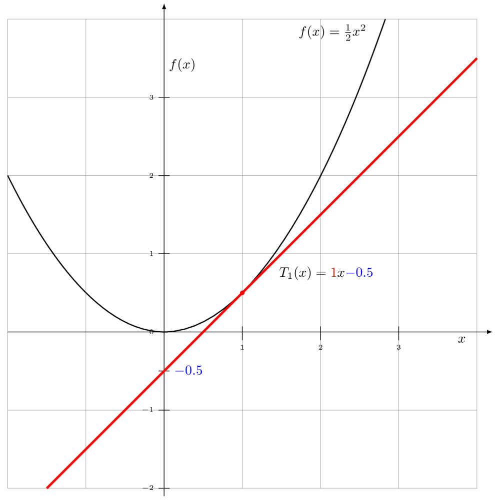
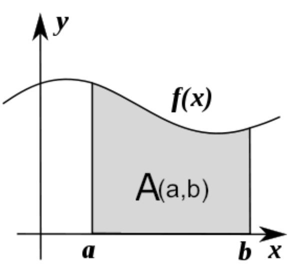


Figure 5.4: An illustration of the tangent line to the function $f(x) = \frac{1}{2}x^2$ at the point $x_0 = 1$. The equation of the tangent line is $T_1(x) = 1x - 0.5$.

the input value x moves farther from x_0, the tangent becomes less accurate at approximating the function $f(x)$.

Integral calculus

The *integral* of $f(x)$ corresponds to the computation of the area under the graph of $f(x)$. The area under $f(x)$ between the points $x = a$ and $x = b$ is denoted as follows:



$$A(a, b) = \int_a^b f(x)\, dx.$$

The area $A(a, b)$ is bounded by the function $f(x)$ from above, by the x-axis from below, and by two vertical lines at $x = a$ and $x = b$. The points $x = a$ and $x = b$ are called the limits of integration. The $\int$ sign comes from the Latin word *summa*. The integral is the "sum" of the values of $f(x)$ between the two limits of integration.

The *integral function* $F(c)$ corresponds to the area calculation as a function of the upper limit of integration:

$$F(c) \equiv \int_0^c f(x)\, dx\,.$$

There are two variables and one constant in this formula. The input variable c describes the upper limit of integration. The *integration variable* x performs a sweep from $x = 0$ until $x = c$. The constant 0 describes the lower limit of integration. Note that choosing $x = 0$ for the starting point of the integral function was an arbitrary choice.

The integral function $F(c)$ contains the "precomputed" information about the area under the graph of $f(x)$. Recall the derivative function $f'(x)$, which tells us the "slope of the graph" property of the function $f(x)$ for all values of x. Similarly, the integral function $F(c)$ tells us the "area under the graph" property of the function $f(x)$ for *all* possible limits of integration.

The area under $f(x)$ between $x = a$ and $x = b$ is obtained by calculating the *change* in the integral function as follows:

$$A(a,b) = \int_a^b f(x)\,dx = F(b) - F(a).$$

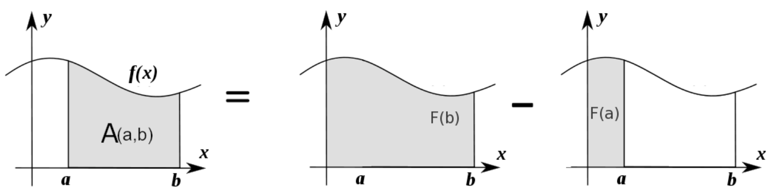


Figure 5.5: The integral function $F(x)$ computes the area under the curve $f(x)$ starting from $x = 0$. The area under $f(x)$ between $x = a$ and $x = b$ is computed using the formula $A(a,b) = F(b) - F(a)$.

Integration techniques

The bulk of the new material needed to understand integral calculus lies in learning various techniques for calculating integrals of functions. Computing integrals is not as easy as computing derivatives, because there are no general rules to follow.

In Section 5.15, we'll describe a number of common techniques for integration. These techniques will enable you to compute the integrals of polynomial functions, exponential functions, logarithmic functions, and trigonometric functions. While these techniques will help you compute integrals in many situations, the process of computing integrals remains somewhat of an art. In art, there are no rules to follow—as an artist, you must be creative and test different approaches until you find one that works.

Applications of integration

Integral calculations have widespread applications to more areas of science than are practical to list here. Let's explore a few examples to gain a general idea of how integrals are applied in the real world.

Computing totals Integral calculations are needed every time we want to compute the total of some quantity that changes over time. If the quantity in question remains constant over time, we can multiply this quantity by the time to find the total quantity. For example, if your monthly rent is \$720, your annual rent is $R = \$720 \times 12$.

But what if your rent changes over time? Imagine a crazy landlord who demands you pay on a daily basis and changes the daily rent $r(t)$ each day. Some days rent is \$20/day, some days \$23/day, and some days he lets you stay for only \$15/day. In this situation, computing your annual rent involves the integral $R = \int_0^{365} r(t)dt$, which describes the calculation of the daily rate $r(t)$ times the duration of each day dt summed over all the days in the year.

Computing potentials In Section 4.6 we defined the notion of potential energy as the negative of the work done when moving an object against a conservative force. We studied two specific cases: gravitational potential energy $U_g(h) \equiv -\int_0^h \vec{F}_g \cdot d\vec{y} = mgh$, and spring potential energy $U_s(x) \equiv -\int_0^x \vec{F}_s(y) \cdot d\vec{y} = \frac{1}{2}kx^2$. Understanding integrals will allow you to solidify your understanding of the connection between each force $\vec{F}_?(x)$ and its associated potential energy $U_?(x)$.

Computing moments of inertia An object's moment of inertia describes how difficult it is to make the object turn. The moment of inertia is computed as the following integral:

$$I = \int_{\text{obj}} r^2 \, dm.$$

In the mechanics chapter, I asked you to memorize the formulas for $I_{\text{disk}} = \frac{1}{2}mR^2$ and $I_{\text{sphere}} = \frac{2}{5}mR^2$ because it was not yet time to explain the details of integral calculations. After learning about integrals, you'll be able to derive the formulas for I_{disk} and I_{sphere} on your own.

Solving differential equations One of the most important applications of integrals is their ability to "undo" the derivative operation.

Recall Newton's second law $F_{\text{net}}(t) = ma(t)$, which can also be written as

$$\frac{F_{\text{net}}(t)}{m} = a(t) = x''(t) = \frac{d}{dx}\left(\frac{d}{dx}x(t)\right).$$

In Chapter 2 we learned how to use integration to solve for $x(t)$ in special cases where the net force is constant $F_{\text{net}}(t) = F_{\text{net}}$. In this chapter, we'll revisit the procedure for finding $x(t)$, and learn how to calculate the motion of an object affected by an external force that varies over time $F_{\text{net}}(t)$.

Limits

The main new tool we'll use in our study of calculus is the notion of a *limit*. In calculus, we often use limits to describe what happens to mathematical expressions when one variable becomes very large, or alternately becomes very small.

For example, to describe a situation where a number n becomes bigger and bigger, we can say,

$$\lim_{n\to\infty} (\text{expression involving } n).$$

This expression is read, "in the limit as n goes to infinity, expression involving n."

Another type of limit occurs when a small, positive number—for example $\delta > 0$, the Greek letter *delta*—becomes progressively smaller and smaller. The precise mathematical statement that describes what happens when the number δ tends to 0 is

$$\lim_{\delta\to 0} (\text{expression involving } \delta),$$

which is read as, "the limit as δ goes to zero, expression involving δ."

Derivative and integral operations are both defined in terms of limits, so understanding limits is essential for calculus. We'll explore limits in more detail and discuss their properties in Section 5.4.

Sequences

So far, we've studied functions defined for real-valued inputs $x \in \mathbb{R}$. We can also study functions defined for natural number inputs $n \in \mathbb{N}$. These functions are called *sequences*.

A sequence is a function of the form $a : \mathbb{N} \to \mathbb{R}$. The sequence's input variable is usually denoted n or k, and it corresponds to the *index* or number in the sequence. We describe sequences either by

specifying the formula for the n^{th} term in the sequence or by listing all the values of the sequence:

$$a_n, n \in \mathbb{N} \quad \Leftrightarrow \quad (a_0, a_1, a_2, a_3, a_4, \ldots).$$

Note the new notation for the input variable as a subscript. This is the standard notation for describing sequences. Also note the sequence continues indefinitely.

An example of a sequence is

$$a_n = \frac{1}{n^2}, \; n \in \mathbb{N}_+ \quad \Leftrightarrow \quad \left(\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right).$$

This sequence is only defined for strictly positive natural numbers $\mathbb{N}_+ = \{1, 2, 3, 4, \ldots\}$ as the input $n = 0$ yields a divide-by-zero error.

The fundamental question we can ask about sequences is whether they *converge* in the limit when n goes to infinity. For instance, the sequence $a_n = \frac{1}{n^2}$ converges to 0 as n goes to infinity. We can express this fact with the limit expression $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

We'll discuss sequences in more detail in Section 5.18.

Series

Suppose we're given a sequence a_n and we want to compute the sum of all the values in this sequence.

To describe the sum of 3^{rd}, 4^{th}, and 5^{th} elements of the sequence a_n, we turn to summation notation:

$$a_3 + a_4 + a_5 \equiv \sum_{3 \leq n \leq 5} a_n \equiv \sum_{n=3}^{5} a_n.$$

The capital Greek letter *sigma* stands in for the word *sum*, and the range of index values included in this sum is denoted below and above the summation sign.

The partial sum of the sequence values a_n ranging from $n = 0$ until $n = N$ is denoted as

$$S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + a_2 + \cdots + a_{N-1} + a_N.$$

The *series* $\sum a_n$ is the sum of *all* the values in the sequence a_n:

$$\sum a_n \equiv S_\infty = \lim_{N\to\infty} S_N = \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots.$$

Note this is an infinite sum.

Techniques

The main mathematical question we'll study with series is the question of their convergence. We say a series $\sum a_n$ *converges* if the infinite sum $S_\infty \equiv \sum_{n\in\mathbb{N}} a_n$ equals some finite number $L \in \mathbb{R}$.

$$S_\infty = \sum_{n=0}^{\infty} a_n = L \quad \Rightarrow \quad \text{the series } \sum a_n \text{ converges.}$$

We call L the *limit* of the series $\sum a_n$.

If the infinite sum $S_\infty \equiv \sum_{n\in\mathbb{N}} a_n$ grows to infinity, we say the series $\sum a_n$ *diverges*.

$$S_\infty = \sum_{n=0}^{\infty} a_n = \pm\infty \quad \Rightarrow \quad \text{the series } \sum a_n \text{ diverges.}$$

The main series technique you need to learn is how to spot the differences between series that converge and series that diverge. You'll learn how to perform different *convergence tests* on the terms in the series, which will indicate whether the infinite sum converges or diverges.

Applications

Series are a powerful computational tool. We can use series to compute approximations to numbers and functions.

For example, the number e can be computed as the following series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3\cdot 2} + \frac{1}{5\cdot 4\cdot 3\cdot 2} + \cdots.$$

The factorial operation $n!$ is the product of n times all integers smaller than n: $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. As we compute more terms from the series, our estimate of the number e becomes more accurate. The partial sum of the first six terms (as shown above) gives us an approximation of e that is accurate to three decimals. The partial sum of the first 12 terms gives us e to an accuracy of nine decimals.

Another useful thing you can do with series is approximate functions by infinitely long polynomials. The *Taylor series* approximation for a function $f(x)$ is defined as the series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Each term in the series is of the form $a_n = c_n x^n$, where c_n is a constant that depends on the function $f(x)$.

For example, the power series of $\sin(x)$ is

$$\sin(x) = \underbrace{\overbrace{x}^{T_1(x)} - \frac{x^3}{3!} + \frac{x^5}{5!}}_{T_5(x)} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots.$$

We can truncate the infinite series anywhere to obtain an approximation to the function. The function $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is the best approximation to the function $\sin(x)$ by a polynomial of degree 5. The equation of the tangent line $T_1(x)$ at $x = 0$ is a special case of the Taylor series approximation procedure, which approximates the function as a first-degree polynomial. We will continue the discussion on series, their properties, and their applications in Section 5.19.

If you haven't noticed yet from glancing at the examples so far, the common theme underpinning all the topics of calculus is the notion of *infinity*. We now turn our attention to the infinite.

5.3 Infinity

Working with infinitely small quantities and infinitely large quantities can be tricky business. It is important that you develop an intuitive understanding of these concepts as soon as possible. Like, now.

Infinitely large

The number ∞ is *really* large. How large? Larger than any number you can think of. Think of any number n. It is true that $n < \infty$. Now think of a bigger number N. It will still hold true that $N < \infty$. In fact, any finite number you can think of, no matter how large, will always be less than ∞.

Technically speaking, ∞ is not a number; infinity is a *process*. You can think of ∞ as the answer you obtain by starting from 0 and continuously adding 1 *forever*.

To see why $N < \infty$ for any finite number N, consider the following reasoning. When we add 1 to a number, we obtain a larger number. The operation $+1$ is equivalent to taking one unit step to the right on the number line. For any n, $n < n+1$. To get to infinity we start from $n = 0$ and keep adding 1. After N steps, we'll arrive at $n = N$. But then we must continue adding 1 and obtain $N+1$, $N+2$, $N+3$, and so on. Since adding 1 always creates a larger number, the following chain of inequalities is true:

$$N < N+1 < N+2 < N+3 < \cdots < \infty.$$

Therefore $N < \infty$ for any finite N.

5.6 Derivatives

In the beginning of the chapter we introduced the derivative concept by identifying the derivative with the slope of the function's graph. This graphical representation of derivatives and the intuition that comes with it are very important: this is how mathematicians and physicists usually "think" about derivatives. It is equally important to understand the formal definition of the derivative operation, so this is what we will cover next. Afterward, we'll build some practical skills for calculating derivatives of functions.

Definition

The derivative of a function is defined as

$$f'(x) \equiv \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}.$$

The definition of the derivative comes from the rise-over-run formula for calculating the slope of a line:

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_f - y_i}{x_f - x_i} = \frac{f(x+\delta) \;-\; f(x)}{x+\delta \;-\; x}.$$

By making δ tend to zero in the above expression, we are able to obtain the slope of the function $f(x)$ at the point x.

Derivatives occur so often in math that people have devised many ways to denote them. Don't be fooled by this multitude of notations—all of them refer to the same concept:

$$Df(x) \equiv f'(x) \equiv \frac{d}{dx}f(x) \equiv \frac{df}{dx} \equiv \dot{f} \equiv \nabla f.$$

Example Let's calculate the derivative of $f(x) = 2x^2 + 3$ to illustrate how the complicated-looking derivative formula works:

$$f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} = \lim_{\delta \to 0} \frac{2(x+\delta)^2 + 3 \;-\; (2x^2+3)}{\delta}.$$

We can simplify the fraction inside the limit:

$$\frac{2x^2 + 4x\delta + \delta^2 - 2x^2}{\delta} = \frac{4x\delta + \delta^2}{\delta} = \frac{4x\delta}{\delta} + \frac{\delta^2}{\delta} = 4x + \delta.$$

The second term of this expression disappears when we take the limit to obtain the final answer:

$$f'(x) = \lim_{\delta \to 0} (4x + \delta) = 4x + 0 = 4x.$$

5.12 Integrals

We now begin our discussion of integrals, the second topic in calculus. An integral is a fancy way of computing the area under the graph of a function. Integral calculus is usually taught as a separate course after differential calculus, but this separation can be counter-productive. The easiest way to understand integration is to think of it as the inverse of the derivative operation. Integrals are antiderivatives. Once you realize this fundamental fact, you'll be able to apply all your differential calculus knowledge to the domain of integral calculus. In differential calculus, we learned how to take a function $f(x)$ and find its derivative $f'(x)$. In integral calculus, we'll be given a function $f(x)$ and we'll be asked to find its *antiderivative* function $F(x)$. The antiderivative of $f(x)$ is a function $F(x)$ whose derivative equals $f(x)$.

In this section, we'll learn about two tasks: how to compute antiderivatives, and how to compute the area under the graph of $f(x)$. Confusingly, both of these tasks are called *integration*. To avoid any possibility of confusion, we'll define the two concepts right away:

- The *indefinite integral* of $f(x)$ is denoted $\int f(x)dx = F(x) + C$. To compute the indefinite integral of $f(x)$, you must find a function $F : \mathbb{R} \to \mathbb{R}$, such that $F'(x) = f(x)$. The indefinite integral is the antiderivative function.

- The *definite integral* of $f(x)$ between $x = a$ and $x = b$ is denoted $\int_a^b f(x)dx = A(a, b)$. Definite integrals correspond to the computation of the area under the function $f(x)$ between $x = a$ and $x = b$. The definite integral is a number $A(a, b) \in \mathbb{R}$.

The two integration tasks are related. The area under the curve $A(a, b)$ can be computed as the *change* in the antiderivative function, using to the formula $A(a, b) = \big[F(x) + C\big]_a^b = F(b) - F(a)$.

Definitions

You should already be familiar with these concepts:

- $\mathbb{R}$: the set of real numbers
- $f(x)$: a function of the form $f : \mathbb{R} \to \mathbb{R}$, which means f takes real numbers as inputs and produces real numbers as outputs
- $\lim_{\delta\to 0}$: a limit expression in which the number δ tends to zero
- $f'(x)$: the derivative of $f(x)$ is the rate of change of f at x:

$$f'(x) = \lim_{\delta\to 0} \frac{f(x+\delta) \; - \; f(x)}{\delta} \,.$$

  The derivative is a function of the form $f' : \mathbb{R} \to \mathbb{R}$.

5.14 The fundamental theorem of calculus

In Section 5.12 we defined the integral function $A_0(x)$ that corresponds to the calculation of the area under $f(x)$ starting from $x = 0$:

$$A_0(x) \equiv \int_0^x f(u)\, du.$$

We also discussed the notion of an antiderivative function: the function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

A priori, there is no reason to suspect the integral function would be related to the derivative operation. The integral corresponds to the computation of an area, whereas the derivative operation computes the slope of a function. The fundamental theorem of calculus describes the relationship between derivatives and integrals.

Theorem (fundamental theorem of calculus)**.** *Let $f(x)$ be a continuous function on the interval $[a, b]$, and let $c \in \mathbb{R}$ be a constant. Define the function $A_c(x)$ as follows:*

$$A_c(x) \equiv \int_c^x f(u)\, du.$$

Then, the derivative of $A_c(x)$ is equal to $f(x)$:

$$\frac{d}{dx}\big[A_c(x)\big] = f(x),$$

for any $x \in (a, b)$.

The fundamental theorem of calculus establishes an equivalence between the set of integral functions and the set of antiderivative functions:

$$A_c(x) = F(x) + C.$$

All integral functions $A_c(x)$ are antiderivatives of $f(x)$.

Differential calculus and integral calculus are two sides of the same coin. If you understand why the theorem is true, you will understand something very deep about calculus. Differentiation is the inverse operation of integration. Given a function $G(x) = \int g(x)dx$, we can obtain the function $g(x)$ by taking the derivative of $G(x)$: $G'(x) = g(x)$. The inverse relationship works in the other direction as well. If you're given the derivative $h'(x)$ of some unknown function $h(x)$, you can find the function $h(x)$ (up to a constant), using integration: $h(x) + C = \int h'(x)dx$.

Got proof?

There is an unspoken rule in mathematics: when the word *theorem* appears in writing, it must be followed by the word *proof*. We therefore need to look into the proof of the fundamental theorem of calculus (FTC). It is not so important you understand the details of the proof, but I still recommend you read this subsection for your general math knowledge. If you are in a rush though, feel free to skip ahead.

Before we get to the proof of the FTC, we'll first introduce the *squeezing principle*, which we'll use in the proof. Suppose you have three functions, f, ℓ, and u, such that

$$\ell(x) \leq f(x) \leq u(x) \qquad \text{for all } x.$$

We say $\ell(x)$ is a *lower bound* on $f(x)$ since its graph is always below that of $f(x)$. Similarly, $u(x)$ is an *upper bound* on $f(x)$. We know the value of $f(x)$ is between $\ell(x)$ and $u(x)$.

Suppose $u(x)$ and $\ell(x)$ both converge to the same limit L:

$$\lim_{x\to a} \ell(x) = L, \quad \text{and} \quad \lim_{x\to a} u(x) = L.$$

Then it must be true that $f(x)$ also converges to the same limit:

$$\lim_{x\to a} f(x) = L.$$

This is true because the function f is *squeezed* between ℓ and u; it has no other choice than to converge to the same limit.

Proof of the fundamental theorem of calculus

For the sake of concreteness, let's use a fixed lower limit of integration $c = 0$. Our starting point is the graph of the function $f(x)$ and the definition of the integral function

$$A_0(x) \equiv \int_0^x f(u)\, du.$$

Our goal is to show that the derivative of the function $A_0(x)$ with respect to x is the function $f(x)$.

Recall the definition of the derivative $g'(x) = \lim_{\epsilon\to 0} \frac{g(x+\epsilon)-g(x)}{\epsilon}$. If we want to find the derivative of $A_0(x)$, we must compute the difference $A_0(x+\epsilon) - A_0(x)$ and then divide by ϵ. Using the definition of the integral function $A_0(x)$, we obtain

$$\begin{aligned} A_0(x+\epsilon) - A_0(x) &= \int_0^{x+\epsilon} f(t)\, dt - \int_0^x f(t)\, dt \\ &= \int_x^{x+\epsilon} f(t)\, dt. \end{aligned}$$

5.15 Techniques of integration

The operation of "taking the integral" of some function is usually much more complicated than that of taking the derivative. You can take the derivative of *any* function—no matter how complex—by using the product rule, the chain rule, and the derivative formulas. This is not true for integrals.

Plenty of integrals have no *closed-form solution*, meaning the function has no antiderivative. There is no simple procedure to follow such that you input a function and "turn the crank" until the integral comes out. Integration is a bit of an art.

Which functions *can* we integrate, and how? Back in the day, scientists collected big tables with integral formulas for various complicated functions. We can use these tables to *look up* a specific integral formula. Such table is given on page 429 in the back of the book.

We can also learn some *integration techniques* to help make complicated integrals simpler. Think of the techniques presented in this section as *adapters*. You can reach for these adapters when the function you need to integrate doesn't appear in your table of integrals, but a similar one is found in the table.

A note to all our students in the audience who are taking an integral calculus course. These integration techniques are exactly the skills you'll be expected to demonstrate on the final. Instead of using the table of integrals to look up complicated integrals, you'll need to know how to fill in the table.

For people interested in learning physics, I'll honestly tell you that if you skip this next section you won't miss much. You should read the important section on *substitution*, but there's no need to read the details of all the recipes for integrating things. For most intents and purposes, once you understand what an integral is, you can use a computer to calculate it. A good tool for calculating integrals is the computer algebra system at `live.sympy.org`.

```
>>> integrate( sin(x) )
-cos(x)
>>> integrate( x**2*exp(x) )
x**2*exp(x) - 2*x*exp(x) + 2*exp(x)
```

You can use `SymPy` for all your integration needs.

A comment to those of you reading this book for *general culture*, without the added stress of homework and exams. Consider the next dozen pages as an ethnographic snapshot of the daily life of the undergraduate experience in science. Try to visualize the life of first-year science students, busy integrating things they don't want to integrate for many, long hours. Picture some unlucky science student locked

5.19 Series

Can you compute $\ln(2)$ using only a basic calculator with four operations, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, and $\boxed{\div}$? I can tell you one way to do this; compute the following infinite sum:

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots .$$

Since the sum is infinite, it will take a while to obtain the value of $\ln(2)$, but if you keep adding more terms in the sum, you will eventually obtain the answer $\ln(2) = 0.693147\ldots$.

Let's make the computer carry out the summation for us. First we define the formula for the n^{th} term in the series $a_n = \frac{(-1)^{n+1}}{n}$, then we compute the sum of the first 100, 1000, and 1000000 terms:

```
>>> def an_ln2(n): return 1.0*(-1)**(n+1)/n
>>> sum([ an_ln2(n)  for n in range(1,100) ])
0.69...
>>> sum([ an_ln2(n)  for n in range(1,1000) ])
0.693...
>>> sum([ an_ln2(n)  for n in range(1,1000000) ])
0.693147...
```

Observe how the approximation becomes more accurate as more terms are added in the sum. A lot of practical mathematical computations are performed in this *iterative* fashion. In this section we'll learn about a powerful technique for calculating quantities to arbitrary precision by summing together more and more terms of a series.

Definitions

- $\mathbb{N} \equiv \{0, 1, 2, 3, 4, 5, 6, \ldots\}$: the set of natural numbers
- $\mathbb{N}_+ \equiv \mathbb{N} \setminus \{0\} \equiv \{1, 2, 3, 4, 5, 6, \ldots\}$: the set of positive natural numbers
- a_n: a sequence of numbers $(a_0, a_1, a_2, a_3, a_4, \ldots)$
- $\sum$: sum. This symbol indicates taking the sum of several objects grouped together. The summation sign is the short way to express certain long expressions:

$$a_3 + a_4 + a_5 + a_6 + a_7 = \sum_{3 \leq n \leq 7} a_n = \sum_{n=3}^{7} a_n .$$

- $\sum a_n$: the series a_n is the sum of all terms in the sequence a_n:

$$S_\infty = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots .$$

- $n!$: the *factorial* function $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$, if $n \geq 1$. We define $0! = 1$.
- $f(x) = \sum_{n=0}^{\infty} c_n x^n$: the *Taylor series* approximation of the function $f(x)$. It has the form of an infinitely long polynomial $c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + \ldots$ where the coefficients c_n are chosen so as to encode the properties of the function $f(x)$.

Exact sums

Formulas exist for calculating the exact sum of certain series.

The sum of the geometric series of length N is

$$S_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + \cdots + r^N = \frac{1 - r^{N+1}}{1 - r}.$$

If $|r| < 1$, taking the limit $N \to \infty$ in the above expression leads to

$$S_\infty = \lim_{N\to\infty} S_N = \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

Example Consider the geometric series with $r = \frac{1}{2}$. Applying the above formula, we obtain

$$S_\infty = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

You can visualize this infinite summation graphically. Imagine starting with a piece of paper of size one-by-one, then adding next to it a second piece of paper with half the size of the first, and a third piece with half the size of the second, and so on. The total area occupied by these pieces of papers is shown in Figure 5.13.

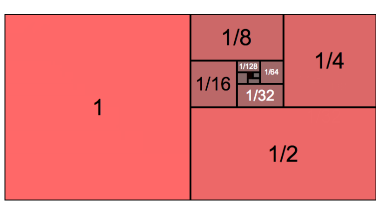


Figure 5.13: A graphical representation of the infinite sum of the geometric series with $r = \frac{1}{2}$. The area of each region corresponds to one of the terms in the series. The total area is equal to $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$.

We'll now state without proof a number of other formulas where the sum of a series can be obtained as a closed-form expression.

5.21 Calculus problems

In this chapter we learned about derivatives and integrals, which are mathematical operations relating to the slope of a function and the area under the graph of a function. We also learned about limits, sequences, and series. It's now time to see how much you've really learned by trying to solve some calculus problems.

Calculus hasn't changed much in the last hundred years. It is testament to this fact that many of the problems presented here were adapted from the book "Calculus Made Easy" by Silvanus Thompson, originally published[1] in 1910. These problems remain as pertinent and interesting today as they were 100 years ago.

As much as calculus is about understanding things conceptually and seeing the big picture (abstraction), calculus is also about practice. There are more than 100 problems to solve in this section. The goal is to turn differentiation and integration into routine operations that you can carry out without stressing out. You should vanquish as many problems as you need to feel comfortable with the procedures of calculus.

Okay, enough prep talk. Let's get to the problems!

Limits problems

P5.1 Use the graph of the function $f(x)$ shown in Figure 5.14 to calculate the following limit expressions:

(1) $\lim_{x\to -5^-} f(x)$ (2) $\lim_{x\to -5^+} f(x)$ (3) $\lim_{x\to -5} f(x)$

(4) $\lim_{x\to 2^-} f(x)$ (5) $\lim_{x\to 2^+} f(x)$ (6) $\lim_{x\to 2} f(x)$

(7) $\lim_{x\to 5^-} f(x)$ (8) $\lim_{x\to 5^+} f(x)$ (9) $\lim_{x\to 5} f(x)$

(10) Is the function $f(x)$ continuous at $x = 5$?

(11) What are the intervals where the function $f(x)$ is continuous?

P5.2 Find the value of the following limit expressions:

(a) $\lim_{x\to 3} 4$ (b) $\lim_{x\to 3} 2x$ (c) $\lim_{x\to 3} x^2 - 2x + 2$

P5.3 Prove the following limit statement by constructing an ϵ, δ-proof:

$$\lim_{x\to 5} 3x = 15.$$

Recall the ϵ,δ-game: one player (the sceptic) specifies the required precision $\epsilon > 0$, and the other player (the prover) must find a value $\delta > 0$ such that $|3x - 15| \leq \epsilon$ for all x in the interval $(5 - \delta, 5 + \delta)$.
Hint: Choose δ to be a multiple of ϵ.

[1] Full text is available at `gutenberg.org/ebooks/33283` (public domain).

End matter

Conclusion

We managed to cover a lot of ground, explaining many topics and concepts in a relatively small textbook. We reviewed high school math and learned about mechanics and calculus. Above all, we examined math and physics material in an integrated manner.

If you liked or hated this book, be sure to send me feedback. Feedback is crucial so I know how to adjust the writing, the content, and the attitude of the book for future learners of math. Please take the time to drop me a line if you find a mistake or to let me know what you thought. You can reach me by email at `ivan.savov@gmail.com`.

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Acknowledgments

This book would not have been possible without the support and encouragement of the people around me. I am fortunate to have grown up surrounded by good people who knew the value of math and encouraged me in my studies and with this project. In this section, I want to *big up* all the people who deserve it.

First and foremost in this list are my parents from whom I have learned many things, and who have supported me throughout my life.

Next in line are all my teachers. I thank my CEGEP teachers: Karnig Bedrossian from whom I learned calculus, Paul Kenton from whom I learned how to think about physics in a chill manner, and Benoit Larose who taught me that more dimensions does not mean things get more complicated. I thank Kohur Gowrisankaran, Frank Ferrie, Mourad El-Gamal, and Ioannis Psaromiligkos for their teach-

Appendix A

Answers and solutions

Chapter 1 solutions

Answers to exercises

E1.1 $x = 2$, $y = 3$. **E1.2** $x = 5$, $y = 6$, and $z = -3$. **E1.3** $p = 7$ and $q = 3$.

Answers to problems

P1.1 $x = \pm 4$. **P1.2** $x = A\cos(\omega t + \phi)$. **P1.3** $x = \frac{ab}{a+b}$. **P1.4** (1) 2.2795. (2) 1024. (3) -8.373. (4) 11. **P1.5** (1) $\frac{3}{4}$. (2) $\frac{-141}{35}$. (3) $3\frac{23}{32}$. **P1.6** (1) c. (2) 1. (3) $\frac{9a}{b}$. (4) a. (5) $\frac{b}{ac}$. (6) $x^2 + ab$. **P1.7** (1) $x^2 + (a-b)x - ab$. (2) $2x^2 - 7x - 15$. (3) $10x^2 + 31x - 14$. **P1.8** (1) $(x-4)(x+2)$. (2) $3x(x-3)(x+3)$. (3) $(x+3)(6x-7)$. **P1.9** (1) $(x-2)^2 + 3$. (2) $2(x+3)^2 + 4$. (3) $6\left(x + \frac{11}{12}\right)^2 - \frac{625}{24}$. **P1.10** \$0.05. **P1.11** 13 people, 30 animals. **P1.12** 5 years later. **P1.13** girl = 80 nuts, boy = 40 nuts. **P1.14** Alice is 15. **P1.15** 18 days. **P1.16** After 2 hours. **P1.18** $\varphi = \frac{1+\sqrt{5}}{2}$. **P1.19** $x = \frac{-5\pm\sqrt{41}}{2}$. **P1.20** (1) $x = \sqrt[3]{2}$. (2) $x = (\frac{\pi}{2} + 2\pi n)$ for $n \in \mathbb{Z}$. **P1.21** No real solutions if $0 < m < 8$. **P1.22** (1) e^z. (2) $\frac{x^3 y^{15}}{z^3}$. (3) $\frac{1}{4x^4}$. (4) $\frac{1}{4}$. (5) -3. (6) $\ln(x+1)$. **P1.23** $\epsilon = 1.110 \times 10^{-16}$; $n = 15.95$ in decimal. **P1.24** (1) $x \in (4, \infty$. (2) $x \in [3, 6]$. (3) $x \in -\infty, -1] \cup [\frac{1}{2}, \infty$. **P1.25** For $n > 250$, Algorithm Q is faster. **P1.26** 10 cm. **P1.27** 22.52 in. **P1.28** $h = \sqrt{3.33^2 - 1.44^2} = 3\,\text{m}$. **P1.29** The opposite side has length 1. **P1.30** $x = \sqrt{3}$, $y = 1$, and $z = 2$. **P1.31** $d = \frac{1800\tan 20^\circ - 800\tan 25^\circ}{\tan 25^\circ - \tan 20^\circ}$, $h = 1658.46\,\text{m}$. **P1.32** $x = \frac{2000}{\tan 24^\circ}$. **P1.33** $x = \tan\theta\sqrt{a^2 + b^2 + c^2}$. **P1.34** $a = \sqrt{3}$, $A_{\triangle} = \frac{3\sqrt{3}}{4}$. **P1.35** $\sin^2\theta\cos^2\theta = \frac{1 - \cos 4\theta}{8}$. **P1.36** $P_{\bigcirc} = 16\tan(22.5^\circ)$, $A_{\bigcirc} = 8\tan(22.5^\circ)$. **P1.37** $c = \frac{a\sin 75^\circ}{\sin 41^\circ} \approx 14.7$. **P1.38** (a) $h = a\sin\theta$. (b) $A = \frac{1}{2}ba\sin\theta$. (c) $c = \sqrt{a^2 + b^2 - 2ab\cos(180 - \theta)}$. **P1.39** $B = 44.8^\circ$, $C = 110.2^\circ$. $c = \frac{a\sin 110.2^\circ}{\sin 25^\circ} \approx 39.97$. **P1.40** $v = 742.92$ km/h. **P1.41** 1.33 cm. **P1.42** $x = 9.55$. **P1.43** $\frac{1}{2}(\pi 4^2 - \pi 2^2) = 18.85$ cm^2. **P1.44** $\ell_{\text{rope}} = 7.83\,\text{m}$. **P1.45** $A_{\text{rect}} = 5c + 10$. **P1.46** $V_{\text{box}} = 1.639\,\text{L}$. **P1.47** $\theta = 120^\circ$. **P1.48** $\frac{R}{r} = \frac{1 - \sin 15^\circ}{\sin 15^\circ} = 2.8637$. **P1.49** 7 cm. **P1.50** $V = 300\,000\,\text{L}$. **P1.51** 315 000 L. **P1.52** 4000 L. **P1.53** $d = \frac{1}{2}(35 - 5\sqrt{21})$. **P1.54** A rope of length $\sqrt{2}\ell$. **P1.55** 20 L of water. **P1.56** $h = 7.84375$ inches. **P1.57**

$1 + 2 + \cdots + 100 = 50 \times 101 = 5050$. **P1.58** $x = -2$ and $y = 2$. **P1.59** $x = 1$, $y = 2$, and $z = 3$. **P1.60** \$112. **P1.61** 20%. **P1.62** \$16501.93. **P1.64** 0.14 s. **P1.65** $\tau = 34.625$ min, 159.45 min. **P1.66** $V(0.01) = 15.58$ volts. $V(0.1) = 1.642$ volts.

Solutions to selected problems

P1.5 For (3), $1\frac{3}{4} + 1\frac{31}{32} = \frac{7}{4} + \frac{63}{32} = \frac{56}{32} + \frac{63}{32} = \frac{119}{32} = 3\frac{23}{32}$.

P1.9 The solutions for (1) and (2) are fairly straightforward. To solve (3), we first factor out 6 from the first two terms to obtain $6(x^2 + \frac{11}{6}x) - 21$. Next we choose half of the coefficient of the linear term to go inside the square and add the appropriate correction to maintain equality: $6[x^2 + \frac{11}{6}x] - 21 = 6[(x + \frac{11}{12})^2 - \left(\frac{11}{12}\right)^2] - 21$. After expanding the rectangular brackets and simplifying, we obtain the final expression: $6\left(x + \frac{11}{12}\right)^2 - \frac{625}{24}$.

P1.11 Let p denote the number of people and a denote the number of animals. We are told $p + a = 43$ and $a = p + 17$. Substituting the second equation into the first, we find $p + (p + 17) = 43$, which is equivalent to $2p = 26$ or $p = 13$. There are 13 figures of people and 30 figures of animals.

P1.12 We must solve for x in $35 + x = 4(5 + x)$. We obtain $35 + x = 20 + 4x$, then $15 = 3x$, so $x = 5$.

P1.14 Let A be Alice's age and B be Bob's age. We're told $A = B + 5$ and $A + B = 25$. Substituting the first equation into the second we find $(B + 5) + B = 25$, which is the same as $2B = 20$, so Bob is 10 years old. Alice is 15 years old.

P1.15 The first shop can bind $4500/30 = 150$ books per day. The second shop can bind $4500/45 = 100$ books per day. The combined production capacity rate is $150 + 100 = 250$ books per day. It will take $4500/250 = 18$ days to bind the books when the two shops work in parallel.

P1.16 Let x denote the distance the slower plane will travel before the two planes meet. Let t_{meet} denote the time when they meet, as measured from the moment the second plane departs. The slower plane must travel x km in t_{meet} hours, so we have $t_{\text{meet}} = \frac{x}{600}$. The faster plane is 600 km behind when it departs. It must travel a distance $(x + 600)$ km in the same time so $t_{\text{meet}} = \frac{x+600}{900}$. Combining the two equations we find $\frac{x}{600} = \frac{x+600}{900}$. After cross-multiplying we find $900x = 600x + 600^2$, which has solution $x = 1200$ km. The time when the planes meet is $t_{\text{meet}} = 2$ hours after the departure of the second plane.

P1.17 This is a funny nonsensical problem that showed up on a school exam. I'm just checking to make sure you're still here.

P1.21 Using the quadratic formula, we find $x = \frac{m \pm \sqrt{m^2 - 8m}}{4}$. If $m^2 - 8m \geq 0$, the solutions are real. If $m^2 - 8m < 0$, the solutions will be complex numbers. Factoring the expressions and plugging in some numbers, we observe that $m^2 - 8m = m(m - 8) < 0$ for all $m \in (0, 8)$.

P1.23 See `bit.ly/float64prec` for the calculations.

P1.24 For (3), $1\frac{3}{4} + 1\frac{31}{32} = \frac{7}{4} + \frac{63}{32} = \frac{56}{32} + \frac{63}{32} = \frac{119}{32} = 3\frac{23}{32}$.

P1.25 The running time of Algorithm Q grows linearly with the size of the problem, whereas Algorithm P's running time grows quadratically. To find the size of the problem when the algorithms take the same time, we solve $P(n) = Q(n)$, which is $0.002n^2 = 0.5n$. The solution is $n = 250$. For $n > 250$, the linear-time algorithm (Algorithm Q) will take less time.

P1.29 Solve for b in Pythagoras' formula $c^2 = a^2 + b^2$ with $c = \varphi$, and $a = \sqrt{\varphi}$. The triangle with sides 1, $\sqrt{\varphi}$, and φ is called Kepler's triangle.

Appendix B

Notation

This appendix contains a summary of the notation used in this book.

Math notation

Expression	Read as	Used to
a, b, x, y		denote variables
$=$	is equal to	indicate two expressions are equal in value
$\equiv$	is defined as	define a variable in terms of an expression
$a + b$	a plus b	combine lengths
$a - b$	a minus b	find the difference in length
$a \times b \equiv ab$	a times b	find the area of a rectangle
$a^2 \equiv aa$	a squared	find the area of a square of side length a
$a^3 \equiv aaa$	a cubed	find the volume of a cube of side length a
a^n	a exponent n	denote a multiplied by itself n times
$\sqrt{a} \equiv a^{\frac{1}{2}}$	square root of a	find the side length of a square of area a
$\sqrt[3]{a} \equiv a^{\frac{1}{3}}$	cube root of a	find the side of a cube with volume a
$a/b \equiv \frac{a}{b}$	a divided by b	denote parts of a whole
$a^{-1} \equiv \frac{1}{a}$	one over a	denotes division by a
$f(x)$	f of x	denote the output of the function f applied to the input x
f^{-1}	f inverse	denote the inverse function of $f(x)$ if $f(x) = y$, then $f^{-1}(y) = x$
e^x	e to the x	denote the exponential function base e
$\ln(x)$	natural log of x	logarithm base e
a^x	a to the x	denote the exponential function base a
$\log_a(x)$	log base a of x	logarithm base a
θ, ϕ	*theta*, *phi*	denote angles
$\sin, \cos, \tan$	sin, cos, tan	obtain trigonometric ratios
$\%$	percent	denote proportions of a total $a\% \equiv \frac{a}{100}$

Set notation

You don't need a lot of fancy notation to understand mathematics. It really helps, though, if you know a little bit of set notation.

Symbol	Read as	Denotes
$\{\ \dots\ \}$	the set ...	define a sets
$\mid$	such that	describe or restrict the elements of a set
$\mathbb{N}$	the naturals	the set $\mathbb{N} \equiv \{0, 1, 2, 3, \dots\}$
$\mathbb{Z}$	the integers	the set $\mathbb{Z} \equiv \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$
$\mathbb{Q}$	the rationals	the set of fractions of integers
$\mathbb{A}$		the set of algebraic numbers
$\mathbb{R}$		the set of real numbers
$\mathbb{C}$		the set of complex numbers
$\subset$	subset	one set strictly contained in another
$\subseteq$	subset or equal	containment or equality
$\cup$	union	the combined elements from two sets
$\cap$	intersection	the elements two sets have in common
$S \setminus T$	S set minus T	the elements of S that are not in T
$a \in S$	a in S	a is an element of set S
$a \notin S$	a not in S	a is not an element of set S
$\forall x$	for all x	a statement that holds for all x
$\exists x$	there exists x	an existence statement
$\nexists x$	there doesn't exist x	a non-existence statement

An example of a math statement using the above notation is "$x \nexists m, n \in \mathbb{Z}$ such that $\sqrt{2} = \frac{m}{n}$." This means

Complex numbers notation

Expression	Denotes
$\mathbb{C}$	the set of complex numbers $\mathbb{C} \equiv \{a + bi \mid a, b \in \mathbb{R}\}$
i	the unit imaginary number $i \equiv \sqrt{-1}$ or $i^2 = -1$
$\text{Re}\{z\} = a$	real part of $z = a + bi$
$\text{Im}\{z\} = b$	imaginary part of $z = a + bi$
$\lvert z\rvert\angle\varphi_z$	polar representation of $z = \lvert z\rvert \cos\varphi_z + i\lvert z\rvert \sin\varphi_z$
$\lvert z\rvert = \sqrt{a^2 + b^2}$	magnitude of $z = a + bi$
$\varphi_z = \tan^{-1}(b/a)$	phase or argument of $z = a + bi$
$\bar{z} = a - bi$	complex conjugate of $z = a + bi$

Vectors notation

Expression	Denotes
$\vec{v}$	a vector
(v_x, v_y)	vector in component notation
$v_x\hat{\imath} + v_y\hat{\jmath}$	vector in unit vector notation
$\|\vec{v}\|\angle\theta$	vector in length-and-direction notation
$\|\vec{v}\|$	length of the vector $\vec{v}$
θ	angle the vector $\vec{v}$ makes with the x-axis
$\hat{v} \equiv \frac{\vec{v}}{\|\vec{v}\|}$	unit length vector in the same direction as $\vec{v}$
$\vec{u} \cdot \vec{v}$	dot product of the vectors $\vec{u}$ and $\vec{v}$
$\vec{u} \times \vec{v}$	cross product of the vectors $\vec{u}$ and $\vec{v}$

Mechanics notation

Expression	Denotes
$x(t)$	position of an object as a function of time
$v(t)$	velocity of an object as a function of time
$a(t)$	acceleration of an object as a function of time
m	mass of an object
$\vec{F}$	a force
$\vec{N}$	normal force
$\vec{F}_{fs}$	static force of friction
$\vec{F}_{fk}$	kinetic force of friction
$\vec{F}_g \equiv \vec{W}$	gravitational force; the weight of an object
U_g	gravitational potential energy
$\vec{F}_s$	force of a spring
U_s	spring potential energy
$\vec{p}$	momentum of a moving object
K	kinetic energy
$\theta(t)$	angular position of a rotating object over time
$\omega(t)$	angular velocity of an object as a function of time
$\alpha(t)$	angular acceleration of an object as a function of time
I_{obj}	moment of inertia of an object
$\mathcal{T}$	torque
L	angular momentum of a spinning object
K_r	rotational kinetic energy of a spinning object

Calculus notation

Expression	Denotes
∞	infinity
ϵ, δ	the Greek letters *epsilon* and *delta*
$f(x)$	a function of the form $f : \mathbb{R} \to \mathbb{R}$
$\lim\limits_{x\to\infty} f(x)$	limit of $f(x)$ as x goes to infinity
$\lim\limits_{x\to a^+} f(x)$	limit of $f(x)$ as x approaches a from the right
$\lim\limits_{x\to a^-} f(x)$	limit of $f(x)$ as x approaches a from the left
$\lim\limits_{x\to a} f(x)$	limit of $f(x)$ as x goes to a
$f'(x)$	derivative of $f(x)$
$f''(x)$	second derivative of $f(x)$
$\frac{d}{dx}$	derivative operator
$F(x)$	antiderivative function of $f(x)$
$\int f(x)\,dx$	indefinite integral of $f(x)$ (spoiler: $\int f(x)dx \equiv F(x)+C$)
$\int_a^b f(x)\,dx$	definite integral of $f(x)$ between $x=a$ and $x=b$
$F(x)\big\vert_\alpha^\beta$	change in $F(x)$ between α and β: $F(x)\big\vert_\alpha^\beta = F(\beta)-F(\alpha)$
$\sum\limits_{k=1}^{N} s_k$	summation of N terms $s_1 + s_2 + \cdots + s_N$
a_n	sequence $a_n : \mathbb{N} \to \mathbb{R}$, also denoted $(a_0, a_1, a_2, a_3, \ldots)$
$\sum\limits_{n=0}^{\infty} a_n$	series $\sum a_n$, which is the infinite sum of the sequence a_n

Appendix C

Constants, units, and conversion ratios

In this appendix you will find a number of tables of useful information that you might need when solving math and physics problems.

Fundamental constants of Nature

Many of the equations of physics include constants as parameters of the equation. For example, Newton's law of gravitation says that the force of gravity between two objects of mass M and m separated by a distance r is $F_g = \frac{GMm}{r^2}$, where G is Newton's gravitational constant.

Symbol	Value	Units	Name
G	$6.673\,84 \times 10^{-11}$	$\text{m}^3\ \text{kg}^{-1}\text{s}^{-2}$	gravitational constant
g	$9.806\,65 \approx 9.81$	m s^{-2}	Earth free-fall acceleration
m_p	$1.672\,621 \times 10^{-27}$	kg	proton mass
m_e	$9.109\,382 \times 10^{-31}$	kg	electron mass
N_A	$6.022\,141 \times 10^{23}$	mol^{-1}	Avogadro's number
k_B	$1.380\,648 \times 10^{-23}$	J K^{-1}	Boltzmann's constant
R	$8.314\,462\,1$	$\text{J K}^{-1}\ \text{mol}^{-1}$	gas constant $R = N_\text{A} k_\text{B}$
μ_0	$1.256\,637 \times 10^{-6}$	N A^{-2}	permeability of free space
ε_0	$8.854\,187 \times 10^{-12}$	F m^{-1}	permittivity of free space
c	$299\,792\,458$	m s^{-1}	speed of light $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$
e	$1.602\,176 \times 10^{-19}$	C	elementary charge
h	$6.626\,069 \times 10^{-34}$	J s	Planck's constant

Appendix D

SymPy tutorial

Computers can be very useful for dealing with complicated math expressions or when slogging through tedious calculations. Throughout this book we used `SymPy` to illustrate several concepts from math and physics. We'll now review all the math and physics tools available through the `SymPy` command line. Don't worry if you're not a computer person; we'll only discuss concepts we covered in the book, and the computer commands we'll learn are very similar to the math operations you're already familiar with. This section also serves as a final review of the material covered in the book.

Introduction

You can use a computer algebra system (CAS) to compute complicated math expressions, solve equations, perform calculus procedures, and simulate physics systems.

All computer algebra systems offer essentially the same functionality, so it doesn't matter which system you use: there are free systems like `SymPy`, `Magma`, or `Octave`, and commercial systems like `Maple`, `MATLAB`, and `Mathematica`. This tutorial is an introduction to `SymPy`, which is a *symbolic* computer algebra system written in the programming language `Python`. In a symbolic CAS, numbers and operations are represented symbolically, so the answers obtained are exact. For example, the number $\sqrt{2}$ is represented in `SymPy` as the object `Pow(2,1/2)`, whereas in *numerical* computer algebra systems like `Octave`, the number $\sqrt{2}$ is represented as the approximation 1.41421356237310 (a `float`). For most purposes the approximation is okay, but sometimes approximations can lead to problems: `float(sqrt(2))*float(sqrt(2)) = 2.00000000000000044` $\neq$ 2. Because `SymPy` uses exact representations, you'll never run into such problems: `Pow(2,1/2)*Pow(2,1/2)`$=2$.

This tutorial presents many explanations as blocks of code. Be sure to try the code examples on your own by typing the commands into SymPy. It's always important to verify for yourself!

Using SymPy

The easiest way to use SymPy, provided you're connected to the Internet, is to visit http://live.sympy.org. You'll be presented with an interactive prompt into which you can enter your commands—right in your browser.

If you want to use SymPy on your own computer, you must install Python and the python package SymPy. You can then open a command prompt and start a SymPy session using:

```
you@host$ python
Python X.Y.Z
[GCC a.b.c (Build Info)] on platform
Type "help", "copyright", or "license" for more information.
>>> from sympy import *
>>>
```

The >>> prompt indicates you're in the Python shell which accepts Python commands. The command `from sympy import *` imports all the SymPy functions into the current namespace. All SymPy functions are now available to you. To exit the python shell press CTRL+D.

I highly recommend you also install ipython, which is an improved interactive python shell. If you have ipython and SymPy installed, you can start an ipython shell with SymPy pre-imported using the command isympy. For an even better experience, you can try ipython notebook, which is a web frontend for the ipython shell.

Each section in this appendix begins with a python import statement for the functions used in that section. If you use the statement `from sympy import *` in the beginning of your code, you don't need to run these individual import statements, but I've included them so you'll know which SymPy vocabulary is covered in each section.

Fundamentals of mathematics

Let's begin by learning about the basic SymPy objects and the operations we can carry out on them. We'll learn the SymPy equivalents of the math verbs we used in Chapter 1: "to solve" (an equation), "to expand" (an expression), "to factor" (a polynomial).

Numbers

```
>>> from sympy import  sympify, S,  evalf, N
```

In Python, there are two types of number objects: ints and floats.

```
>>> 3
3                              # an int
>>> 3.0
3.0                            # a float
```

Integer objects in Python are a faithful representation of the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Floating point numbers are approximate representations of the reals $\mathbb{R}$. Regardless of its absolute size, a floating point number is only accurate to 16 decimals.

Special care is required when specifying rational numbers, because integer division might not produce the answer you want. In other words, Python will not automatically convert the answer to a floating point number, but instead round the answer to the closest integer:

```
>>> 1/7
0                              # int/int gives int
```

To avoid this problem, you can force float division by using the number 1.0 instead of 1:

```
>>> 1.0/7
0.14285714285714285            # float/int gives float
```

This result is better, but it's still only an approximation of the exact number $\frac{1}{7} \in \mathbb{Q}$, since a float has 16 decimals while the decimal expansion of $\frac{1}{7}$ is infinitely long. To obtain an *exact* representation of $\frac{1}{7}$ you need to create a SymPy expression. You can sympify any expression using the shortcut function S():

```
S('1/7')
1/7                            # = Rational(1,7)
```

Note the input to S() is specified as a text string delimited by quotes. We could have achieved the same result using S('1')/7 since a SymPy object divided by an int is a SymPy object.

Except for the tricky Python division operator, other math operators like addition +, subtraction -, and multiplication * work as you would expect. The syntax ** is used in Python to denote exponentiation:

```
>>> 2**10                     # same as S('2^10')
1024
```

When solving math problems, it's best to work with SymPy objects, and wait to compute the numeric answer in the end. To obtain a numeric approximation of a SymPy object as a float, call its .evalf() method:

```
>>> pi
pi
>>> pi.evalf()
3.14159265358979
```

The method `.n()` is equivalent to `.evalf()`. The global `SymPy` function `N()` can also be used to to compute numerical values. You can easily change the number of digits of precision of the approximation. Enter `pi.n(400)` to obtain an approximation of π to 400 decimals.

Symbols

```
>>> from sympy import Symbol, symbols
```

Python is a civilized language so there's no need to define variables before assigning values to them. When you write `>>> a = 3`, you define a new name `a` and set it to the value `3`. You can now use the name `a` in subsequent calculations.

Most interesting `SymPy` calculations require us to define `symbols`, which are the `SymPy` objects for representing variables and unknowns. For your convenience, when `live.sympy.org` starts, it runs the following commands automatically:

```
>>> from __future__ import division
>>> from sympy import *
>>> x, y, z, t = symbols('x y z t')
>>> k, m, n = symbols('k m n', integer=True)
>>> f, g, h = symbols('f g h', cls=Function)
```

The first statement instructs python to convert `1/7` to `1.0/7` when dividing, potentially saving you from any `int` division confusion. The second statement imports all the `SymPy` functions. The remaining statements define some generic symbols `x`, `y`, `z`, and `t`, and several other symbols with special properties.

Note the difference between the following two statements:

```
>>> x + 2
x + 2                  # an Add expression
>>> p + 2
NameError: name 'p' is not defined
```

The name `x` is defined as a symbol, so `SymPy` knows that `x + 2` is an expression; but the variable `p` is not defined, so `SymPy` doesn't know what to make of `p + 2`. To use `p` in expressions, you must first define it as a symbol:

```
>>> p = Symbol('p')    # the same as p = symbols('p')
>>> p + 2
p + 2                  # = Add(Symbol('p'), Integer(2))
```

You can define a sequence of variables using the following notation:

```
>>> a0, a1, a2, a3 = symbols('a0:4')
```

You can use any name you want for a variable, but it's best if you avoid the letters `Q,C,O,S,I,N` and `E` because they have special uses in `SymPy`: `I` is the unit imaginary number $i \equiv \sqrt{-1}$, `E` is the base of the natural logarithm, `S()` is the `sympify` function, `N()` is used to obtain numeric approximations, and `O` is used for big-`O` notation.

The underscore symbol `_` is a special variable that contains the result of the last printed value. The variable `_` is analogous to the `ans` button on certain calculators, and is useful in multi-step calculations:

```
>>> 3+3
6
>>> _*2
12
```

Expressions

```
>>> from sympy import simplify, factor, expand, collect
```

You define `SymPy` expressions by combining symbols with basic math operations and other functions:

```
>>> expr = 2*x + 3*x - sin(x) - 3*x + 42
>>> simplify(expr)
2*x - sin(x) + 42
```

The function `simplify` can be used on any expression to simplify it. The examples below illustrate other useful `SymPy` functions that correspond to common mathematical operations on expressions:

```
>>> factor( x**2-2*x-8 )
(x - 4)*(x + 2)
>>> expand( (x-4)*(x+2) )
x**2 - 2*x - 8
>>> collect(x**2 + x*b + a*x + a*b, x)
x**2 + (a+b)*x + a*b     # collect terms for diff. pows of x
```

To substitute a given value into an expression, call the `.subs()` method, passing in a python dictionary object `{ key:val, ... }` with the symbol–value substitutions you want to make:

```
>>> expr  = sin(x) + cos(y)
>>> expr
sin(x) + cos(y)
>>> expr.subs({x:1, y:2})
sin(1) + cos(2)
>>> expr.subs({x:1, y:2}).n()
0.425324148260754
```

Note how we used `.n()` to obtain the expression's numeric value.

Solving equations

```
>>> from sympy import solve
```

The function `solve` is the main workhorse in `SymPy`. This incredibly powerful function knows how to solve all kinds of equations. In fact `solve` can solve pretty much *any* equation! When high school students learn about this function, they get really angry—why did they spend five years of their life learning to solve various equations by hand, when all along there was this `solve` thing that could do all the math for them? Don't worry, learning math is *never* a waste of time.

The function `solve` takes two arguments. Use `solve(expr,var)` to solve the equation `expr==0` for the variable `var`. You can rewrite any equation in the form `expr==0` by moving all the terms to one side of the equation; the solutions to $A(x) = B(x)$ are the same as the solutions to $A(x) - B(x) = 0$.

For example, to solve the quadratic equation $x^2 + 2x - 8 = 0$, use

```
>>> solve( x**2 + 2*x - 8, x)
[2, -4]
```

In this case the equation has two solutions so `solve` returns a list. Check that $x = 2$ and $x = -4$ satisfy the equation $x^2 + 2x - 8 = 0$.

The best part about `solve` and `SymPy` is that you can obtain symbolic answers when solving equations. Instead of solving one specific quadratic equation, we can solve all possible equations of the form $ax^2 + bx + c = 0$ using the following steps:

```
>>> a, b, c = symbols('a b c')
>>> solve( a*x**2 + b*x + c, x)
[(-b + sqrt(b**2 - 4*a*c))/(2*a), (-b-sqrt(b**2-4*a*c))/(2*a)]
```

In this case `solve` calculated the solution in terms of the symbols `a`, `b`, and `c`. You should be able to recognize the expressions in the solution—it's the quadratic formula $x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

To solve a *system of equations*, you can feed `solve` with the list of equations as the first argument, and specify the list of unknowns you want to solve for as the second argument. For example, to solve for x and y in the system of equations $x + y = 3$ and $3x - 2y = 0$, use

```
>>> solve([x + y - 3, 3*x - 2*y], [x, y])
{x: 6/5, y: 9/5}
```

The function `solve` is like a Swiss Army knife you can use to solve all kind of problems. Suppose you want to *complete the square* in the expression $x^2 - 4x + 7$, that is, you want to find constants h and k such that $x^2 - 4x + 7 = (x - h)^2 + k$. There is no special "complete the square" function in `SymPy`, but you can call `solve` on the equation $(x - h)^2 + k \ - \ (x^2 - 4x + 7) = 0$ to find the unknowns h and k:

```
>>> h, k = symbols('h k')
>>> solve( (x-h)**2 + k  - (x**2-4*x+7), [h,k] )
[(2, 3)]                                       # so h = 2 and k = 3
>>> ((x-2)**2+3).expand()                      # verify...
x**2 - 4*x + 7
```

Learn the basic SymPy commands and you'll never need to suffer another tedious arithmetic calculation painstakingly performed by hand again!

Rational functions

```
>>> from sympy import together, apart
```

By default, SymPy will not combine or split rational expressions. You need to use together to symbolically calculate the addition of fractions:

```
>>> a, b, c, d = symbols('a b c d')
>>> a/b + c/d
a/b + c/d
>>> together(a/b + c/d)
(a*d + b*c)/(b*d)
```

Alternately, if you have a rational expression and want to divide the numerator by the denominator, use the apart function:

```
>>> apart( (x**2+x+4)/(x+2)  )
x - 1  +  6/(x + 2)
```

Exponentials and logarithms

Euler's constant $e = 2.71828\ldots$ is defined one of several ways,

$$e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \equiv \lim_{\epsilon\to 0}\,(1+\epsilon)^{1/\epsilon} \equiv \sum_{n=0}^{\infty}\frac{1}{n!},$$

and is denoted E in SymPy. Using exp(x) is equivalent to E**x.

The functions log and ln both compute the logarithm base e:

```
>>> log(E**3)    # same as ln(E**3)
3
```

By default, SymPy assumes the inputs to functions like exp and log are complex numbers, so it will not expand certain logarithmic expressions. However, indicating to SymPy that the inputs are positive real numbers will make the expansions work:

```
>>> x, y = symbols('x y')
>>> log(x*y).expand()
log(x*y)
>>> a, b = symbols('a b', positive=True)
>>> log(a*b).expand()
log(a) + log(b)
```

Polynomials

Let's define a polynomial P with roots at $x = 1$, $x = 2$, and $x = 3$:

```
>>> P = (x-1)*(x-2)*(x-3)
>>> P
(x - 1)*(x - 2)*(x - 3)
```

To see the expanded version of the polynomial, call its `expand` method:

```
>>> P.expand()
x**3 - 6*x**2 + 11*x - 6
```

When the polynomial is expressed in it's expanded form $P(x) = x^3 - 6^2 + 11x - 6$, we can't immediately identify its roots. This is why the factored form $P(x) = (x-1)(x-2)(x-3)$ is preferable. To factor a polynomial, call its `factor` method or `simplify` it:

```
>>> P.factor()
(x - 1)*(x - 2)*(x - 3)
>>> P.simplify()
(x - 1)*(x - 2)*(x - 3)
```

Recall that the roots of the polynomial $P(x)$ are defined as the solutions to the equation $P(x) = 0$. We can use the `solve` function to find the roots of the polynomial:

```
>>> roots = solve(P,x)
>>> roots
[1, 2, 3]
# let's check if P equals (x-1)(x-2)(x-3)
>>> simplify( P  -  (x-roots[0])*(x-roots[1])*(x-roots[2]) )
0
```

Equality checking

In the last example, we used the `simplify` function to check whether two expressions were equal. This way of checking equality works because $P = Q$ if and only if $P - Q = 0$. This is the best way to check if two expressions are equal in `SymPy` because it attempts all possible simplifications when comparing the expressions. Below is a list of other ways to check whether two quantities are equal with example cases where they fail:

```
>>> p = (x-5)*(x+5)
>>> q = x**2 - 25
>>> p == q                                          # fail
False
>>> p - q == 0                                      # fail
False
>>> simplify(p - q) == 0
True
>>> sin(x)**2 + cos(x)**2  == 1                     # fail
```

```
False
>>> simplify( sin(x)**2 + cos(x)**2  - 1) == 0
True
```

Trigonometry

```
from sympy import sin, cos, tan, trigsimp, expand_trig
```

The trigonometric functions `sin` and `cos` take inputs in radians:

```
>>> sin(pi/6)
1/2
>>> cos(pi/6)
sqrt(3)/2
```

For angles in degrees, you need a conversion factor of $\frac{\pi}{180}$[rad/°]:

```
>>> sin(30*pi/180)                # 30 deg = pi/6 rads
1/2
```

The inverse trigonometric functions $\sin^{-1}(x) \equiv \arcsin(x)$ and $\cos^{-1}(x) \equiv \arccos(x)$ are used as follows:

```
>>> asin(1/2)
pi/6
>>> acos(sqrt(3)/2)
pi/6
```

Recall that $\tan(x) \equiv \frac{\sin(x)}{\cos(x)}$. The inverse function of $\tan(x)$ is $\tan^{-1}(x) \equiv \arctan(x) \equiv$ `atan(x)`

```
>>> tan(pi/6)
1/sqrt(3)                         # = ( 1/2 )/( sqrt(3)/2 )
>>> atan( 1/sqrt(3) )
pi/6
```

The function `acos` returns angles in the range $[0, \pi]$, while `asin` and `atan` return angles in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Here are some trigonometric identities that `SymPy` knows:

```
>>> sin(x) == cos(x - pi/2)
True
>>> simplify( sin(x)*cos(y)+cos(x)*sin(y) )
sin(x + y)
>>> e = 2*sin(x)**2 + 2*cos(x)**2
>>> trigsimp(e)
2
>>> trigsimp(log(e))
log(2*sin(x)**2 + 2*cos(x)**2)
>>> trigsimp(log(e), deep=True)
log(2)

>>> simplify(sin(x)**4 - 2*cos(x)**2*sin(x)**2 + cos(x)**4)
cos(4*x)/2 + 1/2
```

The function `trigsimp` does essentially the same job as `simplify`.

If instead of simplifying you want to expand a trig expression, you should use `expand_trig`, because the default `expand` won't touch trig functions:

```
>>> expand(sin(2*x))          # = (sin(2*x)).expand()
sin(2*x)
>>> expand_trig(sin(2*x))     # = (sin(2*x)).expand(trig=True)
2*sin(x)*cos(x)
```

Complex numbers

```
>>> from sympy import I, re, im, Abs, arg, conjugate
```

Consider the quadratic equation $x^2 = -1$. There are no real solutions to this equation, but we can define an imaginary number $i = \sqrt{-1}$ (denoted `I` in `SymPy`) that satisfies this equation:

```
>>> I*I
-1
>>> solve( x**2 + 1 , x)
[I, -I]
```

The solutions are $x = i$ and $x = -i$, and indeed we can verify that $i^2 + 1 = 0$ and $(-i)^2 + 1 = 0$ since $i^2 = -1$.

The complex numbers $\mathbb{C}$ are defined as $\{a+bi \,|\, a, b \in \mathbb{R}\}$. Complex numbers contain a real part and an imaginary part:

```
>>> z = 4 + 3*I
>>> z
4 + 3*I
>>> re(z)
4
>>> im(z)
3
```

The *polar* representation of a complex number is $z \equiv |z|\angle\theta \equiv |z|e^{i\theta}$. For a complex number $z = a + bi$, the quantity $|z| = \sqrt{a^2 + b^2}$ is known as the absolute value of z, and θ is its *phase* or its *argument*:

```
>>> Abs(z)
5
>>> arg(z)
atan(3/4)
```

The complex conjugate of $z = a + bi$ is the number $\bar{z} = a - bi$:

```
>>> conjugate( z )
4 - 3*I
```

Complex conjugation is important for computing the absolute value of z ($|z| \equiv \sqrt{z\bar{z}}$) and for division by z ($\frac{1}{z} \equiv \frac{\bar{z}}{|z|^2}$).

Euler's formula

```
>>> from sympy import expand, rewrite
```

Euler's formula shows an important relation between the exponential function e^x and the trigonometric functions $\sin(x)$ and $\cos(x)$:

$$e^{ix} = \cos x + i \sin x.$$

To obtain this result in `SymPy`, you must specify that the number x is real and also tell `expand` that you're interested in complex expansions:

```
>>> x = symbols('x', real=True)
>>> exp(I*x).expand(complex=True)
cos(x) + I*sin(x)
>>> re( exp(I*x) )
cos(x)
>>> im( exp(I*x) )
sin(x)
```

Basically, $\cos(x)$ is the real part of e^{ix}, and $\sin(x)$ is the imaginary part of e^{ix}. Whaaat? I know it's weird, but weird things are bound to happen when you input imaginary numbers to functions.

Euler's formula is often used to rewrite the functions sin and cos in terms of complex exponentials. For example,

```
>>> (cos(x)).rewrite(exp)
exp(I*x)/2 + exp(-I*x)/2
```

Compare this expression with the definition of hyperbolic cosine.

Calculus

Calculus is the study of the properties of functions. The operations of calculus are used to describe the limit behaviour of functions, calculate their rates of change, and calculate the areas under their graphs. In this section we'll learn about the `SymPy` functions for calculating limits, derivatives, integrals, and summations.

Infinity

```
from sympy import oo
```

The infinity symbol is denoted `oo` (two lowercase `o`s) in `SymPy`. Infinity is not a number but a process: the process of counting forever. Thus, $\infty + 1 = \infty$, ∞ is greater than any finite number, and $1/\infty$ is an infinitely small number. Sympy knows how to correctly treat infinity in expressions:

```
>>> oo+1
oo
>>> 5000 < oo
True
>>> 1/oo
0
```

Limits

`from sympy import limit`

We use limits to describe, with mathematical precision, infinitely large quantities, infinitely small quantities, and procedures with infinitely many steps.

The number e is defined as the limit $e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$:

```
>>> limit( (1+1/n)**n, n, oo)
E           # = 2.71828182845905
```

This limit expression describes the annual growth rate of a loan with a nominal interest rate of 100% and infinitely frequent compounding. Borrow \$1000 in such a scheme, and you'll owe \$2718.28 after one year.

Limits are also useful to describe the behaviour of functions. Consider the function $f(x) = \frac{1}{x}$. The `limit` command shows us what happens to $f(x)$ near $x = 0$ and as x goes to infinity:

```
>>> limit( 1/x, x, 0, dir="+")
oo
>>> limit( 1/x, x, 0, dir="-")
-oo
>>> limit( 1/x, x, oo)
0
```

As x becomes larger and larger, the fraction $\frac{1}{x}$ becomes smaller and smaller. In the limit where x goes to infinity, $\frac{1}{x}$ approaches zero: $\lim_{x\to\infty}\frac{1}{x} = 0$. On the other hand, when x takes on smaller and smaller positive values, the expression $\frac{1}{x}$ becomes infinite: $\lim_{x\to 0^+}\frac{1}{x} = \infty$. When x approaches 0 from the left, we have $\lim_{x\to 0^-}\frac{1}{x} = -\infty$. If these calculations are not clear to you, study the graph of $f(x) = \frac{1}{x}$.

Here are some other examples of limits:

```
>>> limit(sin(x)/x, x, 0)
1
>>> limit(sin(x)**2/x, x, 0)
0
>>> limit(exp(x)/x**100,x,oo) # which is bigger e^x or x^100 ?
oo                            # exp f >> all poly f for big x
```

Derivatives

The derivative function, denoted $f'(x)$, $\frac{d}{dx}f(x)$, $\frac{df}{dx}$, or $\frac{dy}{dx}$, describes the *rate of change* of the function $f(x)$. The SymPy function diff computes the derivative of any expression:

```
>>> diff(x**3, x)
3*x**2
```

The differentiation operation knows about the product rule $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$, the chain rule $f(g(x))' = f'(g(x))g'(x)$, and the quotient rule $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$:

```
>>> diff( x**2*sin(x), x )
2*x*sin(x) + x**2*cos(x)
>>> diff( sin(x**2),   x )
cos(x**2)*2*x
>>> diff( x**2/sin(x), x )
(2*x*sin(x) - x**2*cos(x))/sin(x)**2
```

The second derivative of a function f is diff(f,x,2):

```
>>> diff(x**3, x, 2)       # same as diff(diff(x**3, x), x)
6*x
```

Tangent lines

The *tangent line* to the function $f(x)$ at $x = x_0$ is the line that passes through the point $(x_0, f(x_0))$ and has the same slope as the function at that point. The tangent line to the function $f(x)$ at the point $x = x_0$ is described by the equation

$$T_1(x) = f(x_0) \ + \ f'(x_0)(x - x_0).$$

What is the equation of the tangent line to $f(x) = \frac{1}{2}x^2$ at $x_0 = 1$?

```
>>> f = S('1/2')*x**2
>>> f
x**2/2
>>> df = diff(f,x)
>>> df
x
>>> T_1 = f.subs({x:1}) + df.subs({x:1})*(x - 1)
>>> T_1
x - 1/2            #  y = x - 1/2
```

The tangent line $T_1(x)$ has the same value and slope as the function $f(x)$ at $x = 1$:

```
>>> T_1.subs({x:1}) == f.subs({x:1})
True
>>> diff(T_1,x).subs({x:1}) == diff(f,x).subs({x:1})
True
```

See Figure 5.6 on page 269.

Optimization

Recall the *second derivative test* for finding the maxima and minima of a function, which we learned on page 284.

Let's find the critical points of the function $f(x) = x^3 - 2x^2 + x$ and use the information from its second derivative to find the maximum of the function on the interval $x \in [0, 1]$.

```
>>> x = Symbol('x')
>>> f = x**3-2*x**2+x
>>> diff(f, x)
3*x**2 - 4*x + 1
>>> sols = solve( diff(f,x),  x)
>>> sols
[1/3, 1]
>>> diff(diff(f,x), x).subs( {x:sols[0]} )
-2
>>> diff(diff(f,x), x).subs( {x:sols[1]} )
2
```

It will help to look at the graph of this function. The point $x = \frac{1}{3}$ is a local maximum because it is a critical point of $f(x)$ where the curvature is negative, meaning $f(x)$ looks like the peak of a mountain at $x = \frac{1}{3}$. The maximum value of $f(x)$ on the interval $x \in [0, 1]$ is $f\left(\frac{1}{3}\right) = \frac{4}{27}$. The point $x = 1$ is a local minimum because it is a critical point with positive curvature, meaning $f(x)$ looks like the bottom of a valley at $x = 1$.

Integrals

In `SymPy` we use `integrate(f, x)` to obtain the integral function $F(x)$ of any function $f(x)$: $F(x) = \int_0^x f(u)\, du$.

```
>>> integrate(x**3, x)
x**4/4
>>> integrate(sin(x), x)
-cos(x)
>>> integrate(ln(x), x)
x*log(x) - x
```

This is known as an *indefinite integral* since the limits of integration are not defined.

In contrast, a *definite integral* computes the area under $f(x)$ between $x = a$ and $x = b$. Use `integrate(f, (x,a,b))` to compute the definite integrals of the form $A(a, b) = \int_a^b f(x)\, dx$:

```
>>> integrate(x**3, (x,0,1))
1/4                  # the area under x^3 from x=0 to x=1
```

We can obtain the same area by first calculating the indefinite integral $F(c) = \int_0^c f(x)\, dx$, then using $A(a, b) = F(x)\Big|_a^b \equiv F(b) - F(a)$:

```
>>> F = integrate(x**3, x)
>>> F.subs({x:1}) - F.subs({x:0})
1/4
```

Integrals correspond to *signed* area calculations:

```
>>> integrate(sin(x), (x,0,pi))
2
>>> integrate(sin(x), (x,pi,2*pi))
-2
>>> integrate(sin(x), (x,0,2*pi))
0
```

During the first half of its 2π-cycle, the graph of $\sin(x)$ is above the x-axis, so it has a positive contribution to the area under the curve. During the second half of its cycle (from $x = \pi$ to $x = 2\pi$), $\sin(x)$ is below the x-axis, so it contributes negative area. Draw a graph of $\sin(x)$ to see what is going on.

Fundamental theorem of calculus

The integral is the "inverse operation" of the derivative. If you perform the integral operation followed by the derivative operation on some function, you'll obtain the same function:

$$\left(\frac{d}{dx} \circ \int dx\right) f(x) \equiv \frac{d}{dx} \int_c^x f(u)\,du = f(x).$$

```
>>> f = x**2
>>> F = integrate(f, x)
>>> F
x**3/3           # + C
>>> diff(F,x)
x**2
```

Alternately, if you compute the derivative of a function followed by the integral, you will obtain the original function $f(x)$ (up to a constant):

$$\left(\int dx \circ \frac{d}{dx}\right) f(x) \equiv \int_c^x f'(u)\,du = f(x) + C.$$

```
>>> f = x**2
>>> df = diff(f,x)
>>> df
2*x
>>> integrate(df, x)
x**2    # + C
```

The fundamental theorem of calculus is important because it tells us how to solve differential equations. If we have to solve for $f(x)$ in the differential equation $\frac{d}{dx}f(x) = g(x)$, we can take the integral on both sides of the equation to obtain the answer $f(x) = \int g(x)\,dx + C$.

Sequences

Sequences are functions that take whole numbers as inputs. Instead of continuous inputs $x \in \mathbb{R}$, sequences take natural numbers $n \in \mathbb{N}$ as inputs. We denote sequences as a_n instead of the usual function notation $a(n)$.

We define a sequence by specifying an expression for its n^{th} term:

```
>>> a_n = 1/n
>>> b_n = 1/factorial(n)
```

Substitute the desired value of n to see the value of the n^{th} term:

```
>>> a_n.subs({n:5})
1/5
```

The `Python` list comprehension syntax `[item for item in list]` can be used to print the sequence values for some range of indices:

```
>>> [ a_n.subs({n:i}) for i in range(0,8) ]
[oo, 1, 1/2, 1/3, 1/4,  1/5,   1/6,   1/7]
>>> [ b_n.subs({n:i}) for i in range(0,8) ]
[1,  1, 1/2, 1/6, 1/24, 1/120, 1/720, 1/5040]
```

Observe that a_n is not properly defined for $n = 0$ since $\frac{1}{0}$ is a division-by-zero error. To be precise, we should say a_n's domain is the positive naturals $a_n : \mathbb{N}^+ \to \mathbb{R}$. Observe how quickly the `factorial` function $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ grows: $7! = 5040$, $10! = 3628800$, $20! > 10^{18}$.

We're often interested in calculating the limits of sequences as $n \to \infty$. What happens to the terms in the sequence when n becomes large?

```
>>> limit(a_n, n, oo)
0
>>> limit(b_n, n, oo)
0
```

Both $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n!}$ *converge* to 0 as $n \to \infty$.

Many important math quantities are defined as limit expressions. An interesting example to consider is the number π, which is defined as the area of a circle of radius 1. We can approximate the area of the unit circle by drawing a many-sided regular polygon around the circle. Splitting the n-sided regular polygon into identical triangular splices, we can obtain a formula for its area A_n (see solution to **P1.36**). In the limit as $n \to \infty$, the n-sided-polygon approximation to the area of the unit-circle becomes exact:

```
>>> A_n = n*tan(2*pi/(2*n))
>>> limit(A_n, n, oo)
pi
```

Series

Suppose we're given a sequence a_n and we want to compute the sum of all the values in this sequence $\sum_n^\infty a_n$. Series are sums of sequences. Summing the values of a sequence $a_n : \mathbb{N} \to \mathbb{R}$ is analogous to taking the integral of a function $f : \mathbb{R} \to \mathbb{R}$.

To work with series in SymPy, use the `summation` function whose syntax is analogous to the `integrate` function:

```
>>> a_n = 1/n
>>> b_n = 1/factorial(n)
>>> summation(a_n, [n, 1, oo])
oo
>>> summation(b_n, [n, 0, oo])
E
```

We say the series $\sum a_n$ *diverges* to infinity (or *is divergent*) while the series $\sum b_n$ converges (or *is convergent*). As we sum together more and more terms of the sequence b_n, the total becomes closer and closer to some finite number. In this case, the infinite sum $\sum_{n=0}^\infty \frac{1}{n!}$ converges to the number $e = 2.71828\ldots$.

The `summation` command is useful because it allows us to compute *infinite* sums, but for most practical applications we don't need to take an infinite number of terms in a series to obtain a good approximation. This is why series are so neat: they represent a great way to obtain approximations.

Using standard Python commands, we can obtain an approximation to e that is accurate to six decimals by summing 10 terms in the series:

```
>>> import math
>>> def b_nf(n):
        return 1.0/math.factorial(n)
>>> sum( [b_nf(n) for n in range(0,10)] )
2.718281 52557319
>>> E.evalf()
2.718281 82845905        # true value
```

Taylor series

Wait, there's more! Not only can we use series to approximate numbers, we can also use them to approximate functions.

A *power series* is a series whose terms contain different powers of the variable x. The n^{th} term in a power series is a function of both the sequence index n and the input variable x.

For example, the power series of the function $\exp(x) = e^x$ is

$$\exp(x) \equiv 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This is, IMHO, one of the most important ideas in calculus: you can compute the value of exp(5) by taking the infinite sum of the terms in the power series with $x = 5$:

```
>>> exp_xn = x**n/factorial(n)
>>> summation( exp_xn.subs({x:5}), [n, 0, oo] ).evalf()
148.413159102577
>>> exp(5).evalf()
148.413159102577          # the true value
```

Note that SymPy is actually smart enough to recognize that the infinite series you're computing corresponds to the closed-form expression e^5:

```
>>> summation( exp_xn.subs({x:5}), [n, 0, oo])
exp(5)
```

Taking as few as 35 terms in the series is sufficient to obtain an approximation to e that is accurate to 16 decimals:

```
>>> import math                    # redo using only python
>>> def exp_xnf(x,n):
        return x**n/math.factorial(n)
>>> sum( [exp_xnf(5.0,i) for i in range(0,35)] )
148.413159102577
```

The coefficients in the power series of a function (also known as the *Taylor series*) depend on the value of the higher derivatives of the function. The formula for the n^{th} term in the Taylor series of $f(x)$ expanded at $x = c$ is $a_n(x) = \frac{f^{(n)}(c)}{n!}(x-c)^n$, where $f^{(n)}(c)$ is the value of the n^{th} derivative of $f(x)$ evaluated at $x = c$. The term *Maclaurin series* refers to Taylor series expansions at $x = 0$.

The SymPy function `series` is a convenient way to obtain the series of any function. Calling `series(expr,var,at,nmax)` will show you the series expansion of `expr` near `var`=`at` up to power `nmax`:

```
>>> series( sin(x), x, 0, 8)
x - x**3/6 + x**5/120 - x**7/5040 + O(x**8)
>>> series( cos(x), x, 0, 8)
1 - x**2/2 + x**4/24 - x**6/720 + O(x**8)
>>> series( sinh(x), x, 0, 8)
x + x**3/6 + x**5/120 + x**7/5040 + O(x**8)
>>> series( cosh(x), x, 0, 8)
1 + x**2/2 + x**4/24 + x**6/720 + O(x**8)
```

Some functions are not defined at $x = 0$, so we expand them at a different value of x. For example, the power series of $\ln(x)$ expanded at $x = 1$ is

```
>>> series(ln(x), x, 1, 6)     # Taylor series of ln(x) at x=1
x - x**2/2 + x**3/3 - x**4/4 + x**5/5  + O(x**6)
```

Here, the result `SymPy` returns is misleading. The Taylor series of $\ln(x)$ expanded at $x = 1$ has terms of the form $(x-1)^n$:

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} + \cdots .$$

Verify this is the correct formula by substituting $x = 1$. `SymPy` returns an answer in terms of coordinates *relative* to $x = 1$.

Instead of expanding $\ln(x)$ around $x = 1$, we can obtain an equivalent expression if we expand $\ln(x+1)$ around $x = 0$:

```
>>> series(ln(x+1), x, 0, 6)   # Maclaurin series of ln(x+1)
x - x**2/2 + x**3/3 - x**4/4 + x**5/5 + O(x**6)
```

Vectors

A vector $\vec{v} \in \mathbb{R}^n$ is an n-tuple of real numbers. For example, consider a vector that has three components:

$$\vec{v} = (v_1, v_2, v_3) \;\in\; (\mathbb{R}, \mathbb{R}, \mathbb{R}) \equiv \mathbb{R}^3.$$

To specify the vector $\vec{v}$, we specify the values for its three components v_1, v_2, and v_3.

A matrix $A \in \mathbb{R}^{m \times n}$ is a rectangular array of real numbers with m rows and n columns. A vector is a special type of matrix; we can think of a vector $\vec{v} \in \mathbb{R}^n$ either as a row vector ($1 \times n$ matrix) or a column vector ($n \times 1$ matrix). Because of this equivalence between vectors and matrices, there is no need for a special vector object in `SymPy`, and `Matrix` objects are used for vectors as well.

This is how we define vectors and compute their properties:

```
>>> u = Matrix([[4,5,6]]) # a row vector = 1x3 matrix
>>> v = Matrix([[7],
                [8],      # a col vector = 3x1 matrix
                [9]])
>>> v.T                   # use the transpose operation to
Matrix([[7, 8, 9]])       # convert a col vec to a row vec

>>> u[0]                  # 0-based indexing for entries
4
>>> u.norm()              # length of u
sqrt(77)
>>> uhat = u/u.norm()     # unit-length vec in same dir as u
>>> uhat
[4/sqrt(77), 5/sqrt(77), 6/sqrt(77)]
>>> uhat.norm()
1
```

Dot product

The dot product of the 3-vectors $\vec{u}$ and $\vec{v}$ can be defined two ways:

$$\vec{u}\cdot\vec{v} \equiv \underbrace{u_xv_x + u_yv_y + u_zv_z}_{\text{algebraic def.}} \equiv \underbrace{\|\vec{u}\|\|\vec{v}\|\cos(\varphi)}_{\text{geometric def.}} \quad \in \mathbb{R},$$

where φ is the angle between the vectors $\vec{u}$ and $\vec{v}$. In SymPy,

```
>>> u = Matrix([ 4,5,6])
>>> v = Matrix([-1,1,2])
>>> u.dot(v)
13
```

We can combine the algebraic and geometric formulas for the dot product to obtain the cosine of the angle between the vectors

$$\cos(\varphi) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{u_xv_x + u_yv_y + u_zv_z}{\|\vec{u}\|\|\vec{v}\|},$$

and use the acos function to find the angle measure:

```
>>> acos(u.dot(v)/(u.norm()*v.norm())).evalf()
0.921263115666387       # in radians  =  52.76 degrees
```

Just by looking at the coordinates of the vectors $\vec{u}$ and $\vec{v}$, it's difficult to determine their relative direction. Thanks to the dot product, however, we know the angle between the vectors is $52.76°$, which means they *kind of* point in the same direction. Vectors that are at an angle $\varphi = 90°$ are called *orthogonal*, meaning at right angles with each other. The dot product of vectors for which $\varphi > 90°$ is negative because they point *mostly* in opposite directions.

The notion of the "angle between vectors" applies more generally to vectors with any number of dimensions. The dot product for n-dimensional vectors is $\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_iv_i$. This means we can talk about "the angle between" 1000-dimensional vectors. That's pretty crazy if you think about it—there is no way we could possibly "visualize" 1000-dimensional vectors, yet given two such vectors we can tell if they point mostly in the same direction, in perpendicular directions, or mostly in opposite directions.

The dot product is a commutative operation $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$:

```
>>> u.dot(v) == v.dot(u)
True
```

Cross product

The *cross product*, denoted $\times$, takes two vectors as inputs and produces a vector as output. The cross products of individual basis elements are defined as follows:

$$\hat{\imath}\times\hat{\jmath} = \hat{k}, \qquad \hat{\jmath}\times\hat{k} = \hat{\imath}, \qquad \hat{k}\times\hat{\imath} = \hat{\jmath}.$$

Here is how to compute the cross product of two vectors in SymPy:

```
>>> u = Matrix([ 4,5,6])
>>> v = Matrix([-1,1,2])
>>> u.cross(v)
[4, -14, 9]
```

The vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. The norm of the cross product $\|\vec{u} \times \vec{v}\|$ is proportional to the lengths of the vectors and the sine of the angle between them:

```
(u.cross(v).norm()/(u.norm()*v.norm())).n()
0.796366206088088     # = sin(0.921..)
```

The name "cross product" is well-suited for this operation since it is calculated by "cross-multiplying" the coefficients of the vectors:

$$\vec{u} \times \vec{v} = \left(u_y v_z - u_z v_y,\ u_z v_x - u_x v_z,\ u_x v_y - u_y v_x\right).$$

By defining individual symbols for the entries of two vectors, we can make SymPy show us the cross-product formula:

```
>>> u1,u2,u3 = symbols('u1:4')
>>> v1,v2,v3 = symbols('v1:4')
>>> Matrix([u1,u2,u3]).cross(Matrix([v1,v2,v3]))
[ (u2*v3 - u3*v2), (-u1*v3 + u3*v1), (u1*v2 - u2*v1) ]
```

The dot product is anti-commutative $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$:

```
>>> u.cross(v)
[4, -14, 9]
>>> v.cross(u)
[-4, 14,-9]
```

The product of two numbers and the dot product of two vectors are commutative operations. The cross product, however, is not commutative: $\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$.

Mechanics

The module called `sympy.physics.mechanics` contains elaborate tools for describing mechanical systems, manipulating reference frames, forces, and torques. These specialized functions are not necessary for a first-year mechanics course. The basic SymPy functions like `solve`, and the vector operations you learned in the previous sections are powerful enough for basic Newtonian mechanics.

Dynamics

The net force acting on an object is the sum of all the external forces acting on it $\vec{F}_{\text{net}} = \sum \vec{F}$. Since forces are vectors, we need to use vector addition to compute the net force.

Compute $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2$, where $\vec{F}_1 = 4\hat{\imath}$[N] and $\vec{F}_2 = 5\angle 30°$[N]:

```
>>> F_1 =  Matrix( [4,0] )
>>> F_2 =  Matrix( [5*cos(30*pi/180), 5*sin(30*pi/180) ] )
>>> F_net = F_1 + F_2
>>> F_net
[4 + 5*sqrt(3)/2,   5/2]          # in Newtons
>>> F_net.evalf()
[8.33012701892219,  2.5]          # in Newtons
```

To express the answer in length-and-direction notation, use `norm` to find the length of $\vec{F}_{\text{net}}$ and `atan2`[1] to find its direction:

```
>>> F_net.norm().evalf()
8.69718438067042                  # |F_net| in [N]
>>> (atan2( F_net[1],F_net[0] )*180/pi).n()
16.7053138060100                  # angle in degrees
```

The net force on the object is $\vec{F}_{\text{net}} = 8.697\angle 16.7°[\text{N}]$.

Kinematics

Let $x(t)$ denote the position of an object, $v(t)$ denote its velocity, and $a(t)$ denote its acceleration. Together $x(t)$, $v(t)$, and $a(t)$ are known as the *equations of motion* of the object.

Starting from the knowledge of $\vec{F}_{\text{net}}$, we can compute $a(t) = \frac{\vec{F}_{\text{net}}}{m}$, then obtain $v(t)$ by integrating $a(t)$, and finally obtain $x(t)$ by integrating $v(t)$:

$$\underbrace{\frac{\vec{F}_{\text{net}}}{m} = a(t)}_{\text{Newton's 2}^{\text{nd}}\text{ law}} \underbrace{\overset{v_i+\int dt}{\longrightarrow} \quad v(t) \quad \overset{x_i+\int dt}{\longrightarrow} \quad x(t).}_{\text{kinematics}}$$

Uniform acceleration motion (UAM)

Let's analyze the case where the net force on the object is constant. A constant force causes a constant acceleration $a = \frac{F}{m} = \text{constant}$. If the acceleration function is constant over time $a(t) = a$. We find $v(t)$ and $x(t)$ as follows:

```
>>> t, a, v_i, x_i = symbols('t v_i x_i a')
>>> v = v_i + integrate(a, (t, 0,t) )
>>> v
a*t + v_i
>>> x = x_i + integrate(v, (t, 0,t) )
>>> x
a*t**2/2 + v_i*t + x_i
```

[1]The function `atan2(y,x)` computes the correct direction for all vectors (x, y), unlike `atan(y/x)` which requires corrections for angles in the range $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

You may remember these equations from Section 2.4 (page 121). They are the *uniform accelerated motion* (UAM) equations:

$$\begin{aligned} a(t) &= a, \\ v(t) &= v_i + at, \\ x(t) &= x_i + v_i t + \frac{1}{2}at^2. \end{aligned}$$

In high school, you probably had to memorize these equations. Now you know how to derive them yourself starting from first principles.

For the sake of completeness, we'll now derive the fourth UAM equation, which relates the object's final velocity to the initial velocity, the displacement, and the acceleration, without reference to time:

```
>>> (v*v).expand()
a**2*t**2 + 2*a*t*v_i + v_i**2
>>> ((v*v).expand() - 2*a*x).simplify()
-2*a*x_i + v_i**2
```

The above calculation shows $v_f^2 - 2ax_f = -2ax_i + v_i^2$. After moving the term $2ax_f$ to the other side of the equation, we obtain

$$(v(t))^2 \;=\; v_f^2 = v_i^2 + 2a\Delta x \;=\; v_i^2 + 2a(x_f - x_i).$$

The fourth equation is important for practical purposes because it allows us to solve physics problems in a time-less manner.

Example

Find the position function of an object at time $t = 3$[s], if it starts from $x_i = 20$[m] with $v_i = 10$[m/s] and undergoes a constant acceleration of $a = 5$[m/s^2]. What is the object's velocity at $t = 3$[s]?

```
>>> x_i = 20   # initial position
>>> v_i = 10   # initial velocity
>>> a   = 5    # acceleration (constant during motion)
>>> x = x_i + integrate(  v_i+integrate(a,(t,0,t)),  (t,0,t) )
>>> x
5*t**2/2 + 10*t + 20
>>> x.subs({t:3}).n()           # x(3) in [m]
72.5
>>> diff(x,t).subs({t:3}).n()   # v(3) in [m/s]
25                              # = sqrt( v_i**2 + 2*a*52.5 )
```

If you think about it, physics knowledge combined with computer skills is like a superpower!

General equations of motion

The procedure $a(t) \overset{v_i+\int dt}{\longrightarrow} v(t) \overset{x_i+\int dt}{\longrightarrow} x(t)$ can be used to obtain the position function $x(t)$ even when the acceleration is not constant. Suppose the acceleration of an object is $a(t) = \sqrt{kt}$; what is its $x(t)$?

```
>>> t, v_i, x_i, k = symbols('t v_i x_i k')
>>> a = sqrt(k*t)
>>> x = x_i + integrate( v_i+integrate(a,(t,0,t)), (t, 0,t) )
>>> x
x_i + v_i*t + (4/15)*(k*t)**(5/2)/k**2
```

Potential energy

For each force $\vec{F}(x)$ there is a corresponding potential energy $U_F(x)$. The change in potential energy associated with the force $\vec{F}(x)$ and displacement $\vec{d}$ is defined as the negative of the work done by the force during the displacement: $U_F(x) = -W = -\int_{\vec{d}} \vec{F}(x) \cdot d\vec{x}$.

The potential energies associated with gravity $\vec{F}_g = -mg\hat{\jmath}$ and the force of a spring $\vec{F}_s = -k\vec{x}$ are calculated as follows:

```
>>> x, y = symbols('x y')
>>> m, g, k, h = symbols('m g k h')
>>> F_g = -m*g                # Force of gravity on mass m
>>> U_g = - integrate( F_g, (y,0,h) )
>>> U_g
m*g*h                         # Grav. potential energy
>>> F_s = -k*x                # Spring force for displacement x
>>> U_s = - integrate( F_s, (x,0,x) )
>>> U_s
k*x**2/2                      # Spring potential energy
```

Note the negative sign in the formula defining the potential energy. This negative is canceled by the negative sign of the dot product $\vec{F} \cdot d\vec{x}$: when the force acts in the direction opposite to the displacement, the work done by the force is negative.

Simple harmonic motion

```
from sympy import Function, dsolve
```

The force exerted by a spring is given by the formula $F = -kx$. If the only force acting on a mass m is the force of a spring, we can use Newton's second law to obtain the following equation:

$$F = ma \quad \Rightarrow \quad -kx = ma \quad \Rightarrow \quad -kx(t) = m\frac{d^2}{dt^2}\Big[x(t)\Big].$$

The motion of a mass-spring system is described by the *differential equation* $\frac{d^2}{dt^2}x(t) + \omega^2 x(t) = 0$, where the constant $\omega = \sqrt{\frac{k}{m}}$ is called

the angular frequency. We can find the position function $x(t)$ using the dsolve method:

```
>>> t = Symbol('t')                    # time               t
>>> x = Function('x')                  # position function x(t)
>>> w = Symbol('w', positive=True)     # angular frequency w
>>> sol = dsolve( diff(x(t),t,t) + w**2*x(t), x(t) )
>>> sol
x(t) == C1*sin(w*t) + C2*cos(w*t)
>>> x = sol.rhs
>>> x
C1*sin(w*t) + C2*cos(w*t)
```

Note the solution $x(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t)$ is equivalent to $x(t) = A\cos(\omega t + \phi)$, which is more commonly used to describe simple harmonic motion. We can use the expand function with the argument trig=True to convince ourselves of this equivalence:

```
>>> A, phi = symbols("A phi")
>>> (A*cos(w*t - phi)).expand(trig=True)
A*sin(phi)*sin(w*t) + A*cos(phi)*cos(w*t)
```

If we define $C_1 = A\sin(\phi)$ and $C_2 = A\cos(\phi)$, we obtain the form $x(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t)$ that SymPy found.

Conservation of energy

We can verify that the total energy of the mass-spring system is conserved by showing $E_T(t) = U_s(t) + K(t) = \text{constant}$:

```
>>> x = sol.rhs.subs({"C1":0,"C2":A})
>>> x
A*cos(t*w)
>>> v = diff(x, t)
-A*w*sin(t*w)
>>> E_T = (0.5*k*x**2 + 0.5*m*v**2).simplify()
>>> E_T
0.5*A**2*(k*cos(w*t)**2 + m*w**2*sin(w*t)**2)
>>> E_T.subs({k:m*w**2}).simplify()
0.5*m*(w*A)**2                                 # = K_max
>>> E_T.subs({w:sqrt(k/m)}).simplify()
0.5*k*A**2                                     # = U_max
```

Conclusion

I'll conclude with some words of caution about computer overuse. Computer technology is very powerful and is everywhere around us, but we must not forget that computers are actually very dumb. Computers are merely calculators, and they depend on your knowledge to direct them. It's important you learn how to perform complicated math by hand in order to be able to instruct computers to execute math for you, and so you can check the results of your computer calculations. I don't want you to use the tricks you learned in this tutorial to avoid math problems and blindly rely on `SymPy` for all your math needs. That won't work! The idea is for both you and the computer to be math powerhouses.

Most math discoveries were made using pen and paper. When solving a math problem, if you clearly define each variable, draw a diagram, and clearly set up the problem's equations in terms of the variables you defined, then half the work of solving the problem is done. Computers can't help with these important, initial modelling and problem-specific tasks—only humans are good at this stuff. Once you *set up* the problem, `SymPy` can help you breeze through tedious calculations. The combination of pen and paper for thinking and `SymPy` for calculating is indeed quite powerful. Go out there and do some science!

Links

[Installation instructions for `ipython notebook`]
`http://ipython.org/install.html`

[The official `SymPy` tutorial]
`http://docs.sympy.org/latest/tutorial/intro.html`

[A list of `SymPy` gotchas]
`http://docs.sympy.org/dev/gotchas.html`

Appendix E

Formulas

Calculus formulas

$\dfrac{dy}{dx}$	$\longleftarrow \quad y \quad \longrightarrow$	$\displaystyle\int y\,dx$
Algebraic		
1	x	$\frac{1}{2}x^2 + C$
0	a	$ax + C$
nx^{n-1}	x^n	$\dfrac{1}{n+1}x^{n+1} + C$
$-x^{-2}$	x^{-1}	$\ln x + C$
$\dfrac{du}{dx} \pm \dfrac{dv}{dx} \pm \dfrac{dw}{dx}$	$u \pm v \pm w$	$\int u\,dx \pm \int v\,dx \pm \int w\,dx$
$u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$	uv	No general form known
$\dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$	$\dfrac{u}{v}$	No general form known
	u	$ux - \int x\,du + C$

$\dfrac{dy}{dx}$	$\longleftarrow \quad y \quad \longrightarrow$	$\displaystyle\int y\,dx$
Exponential and Logarithmic		
e^x	e^x	$e^x + C$
x^{-1}	$\ln x$	$x(\ln x - 1) + C$
$\frac{1}{\ln 10}\, x^{-1}$	$\log_{10} x$	$\frac{1}{\ln 10}\, x(\ln x - 1) + C$
$a^x \ln a$	a^x	$\dfrac{a^x}{\ln a} + C$
Trigonometric		
$\cos x$	$\sin x$	$-\cos x + C$
$-\sin x$	$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x$	$-\ln \cos x + C$
Inverse trigonometric		
$\dfrac{1}{\sqrt{(1-x^2)}}$	$\sin^{-1}(x)$	$x\sin^{-1}(x) + \sqrt{1-x^2} + C$
$-\dfrac{1}{\sqrt{(1-x^2)}}$	$\cos^{-1}(x)$	$x\cos^{-1}(x) - \sqrt{1-x^2} + C$
$\dfrac{1}{1+x^2}$	$\tan^{-1}(x)$	$x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C$
Hyperbolic		
$\cosh x$	$\sinh x$	$\cosh x + C$
$\sinh x$	$\cosh x$	$\sinh x + C$
$\operatorname{sech}^2 x$	$\tanh x$	$\ln\cosh x + C$
Inverse hyperbolic		
$-\dfrac{x}{(a^2+x^2)^{\frac{3}{2}}}$	$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}(\frac{x}{a}) + C \equiv \ln(x + \sqrt{a^2+x^2}) + C$

$\dfrac{dy}{dx}$	$\longleftarrow \quad y \quad \longrightarrow$	$\displaystyle\int y\,dx$
	Miscellaneous	
$-\dfrac{1}{(x+a)^2}$	$\dfrac{1}{x+a}$	$\ln(x+a)+C$
$\mp\dfrac{b}{(a\pm bx)^2}$	$\dfrac{1}{a\pm bx}$	$\pm\dfrac{1}{b}\ln(a\pm bx)+C$
$-\dfrac{3a^2x}{(a^2+x^2)^{\frac{5}{2}}}$	$\dfrac{a^2}{(a^2+x^2)^{\frac{3}{2}}}$	$\dfrac{x}{\sqrt{a^2+x^2}}+C$
$a\cos ax$	$\sin ax$	$-\dfrac{1}{a}\cos ax+C$
$-a\sin ax$	$\cos ax$	$\dfrac{1}{a}\sin ax+C$
$a\sec^2 ax$	$\tan ax$	$-\dfrac{1}{a}\ln\cos ax+C$
$\sin 2x$	$\sin^2 x$	$\dfrac{x}{2}-\dfrac{\sin 2x}{4}+C$
$-\sin 2x$	$\cos^2 x$	$\dfrac{x}{2}+\dfrac{\sin 2x}{4}+C$
$n\sin^{n-1}x\cos x$	$\sin^n x$	$-\dfrac{\cos x}{n}\sin^{n-1}x+\dfrac{n-1}{n}\displaystyle\int\sin^{n-2}x\,dx+C$
$-\dfrac{\cos x}{\sin^2 x}$	$\dfrac{1}{\sin x}$	$\ln\tan\dfrac{x}{2}+C$
$-\dfrac{\sin 2x}{\sin^4 x}$	$\dfrac{1}{\sin^2 x}$	$-\operatorname{cotan} x+C$
$\dfrac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}$	$\dfrac{1}{\sin x\cos x}$	$\ln\tan x+C$
$n\sin mx\cos nx+$ $m\sin nx\cos mx$	$\sin mx\sin nx$	$\frac{1}{2}\cos(m-n)x-\frac{1}{2}\cos(m+n)x+C$
$2a\sin 2ax$	$\sin^2 ax$	$\dfrac{x}{2}-\dfrac{\sin 2ax}{4a}+C$
$-2a\sin 2ax$	$\cos^2 ax$	$\dfrac{x}{2}+\dfrac{\sin 2ax}{4a}+C$

Mechanics formulas

Forces:

$$W = F_g = \frac{GMm}{r^2} = gm, \quad F_s = -kx, \quad F_{fs} \leq \mu_s N, \quad F_{fk} = \mu_k N$$

Newton's three laws:

$$\text{if no } \vec{F}_{\text{ext}}, \text{ then } \vec{v}_i = \vec{v}_f \tag{1}$$
$$\vec{F}_{\text{net}} = m\vec{a} \tag{2}$$
$$\text{if } \vec{F}_{12}, \text{ then } \exists \vec{F}_{21} = -\vec{F}_{12} \tag{3}$$

Uniform acceleration motion (UAM):

$$a(t) = a \tag{4}$$
$$v(t) = at + v_i \tag{5}$$
$$x(t) = \tfrac{1}{2}at^2 + v_i t + x_i \tag{6}$$
$$v_f^2 = v_i^2 + 2a\Delta x \tag{7}$$

Momentum:

$$\vec{p} = m\vec{v} \tag{8}$$

Energy and work:

$$K = \tfrac{1}{2}mv^2, \quad U_g = mgh, \quad U_s = \tfrac{1}{2}kx^2, \quad K_r = \tfrac{1}{2}I\omega^2, \quad W = \vec{F} \cdot \vec{d} \tag{9}$$

Conservation laws:

$$\sum \vec{p}_{\text{in}} = \sum \vec{p}_{\text{out}} \tag{10}$$
$$L_{\text{in}} = L_{\text{out}} \tag{11}$$
$$\sum E_{\text{in}} + W_{\text{in}} = \sum E_{\text{out}} + W_{\text{out}} \tag{12}$$

Circular motion (radial acceleration and radial force):

$$a_r = \frac{v_t^2}{R}, \qquad \vec{F}_r = ma_r\hat{r} \tag{13}$$

Angular motion:

$$F = ma \quad \Rightarrow \quad \mathcal{T} = I\alpha \tag{14}$$
$$a(t), v(t), x(t) \quad \Rightarrow \quad \alpha(t), \omega(t), \theta(t) \tag{15}$$
$$\vec{p} = m\vec{v} \quad \Rightarrow \quad L = I\omega \tag{16}$$
$$K = \tfrac{1}{2}mv^2 \quad \Rightarrow \quad K_r = \tfrac{1}{2}I\omega^2 \tag{17}$$

SHM with $\omega = \sqrt{\frac{k}{m}}$ (mass-spring system) or $\omega = \sqrt{\frac{g}{\ell}}$ (pendulum):

$$x(t) = A\cos(\omega t + \phi) \tag{18}$$
$$v(t) = -A\omega\sin(\omega t + \phi) \tag{19}$$
$$a(t) = -A\omega^2\cos(\omega t + \phi) \tag{20}$$

Often calculus and mechanics are taught as separate subjects. It shouldn't be like that. Learning calculus without mechanics is incredibly boring. Learning mechanics without calculus is missing the point. This textbook integrates both subjects and highlights the profound connections between them.

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The author has more than 14 years of tutoring experience, a B.Eng. in electrical engineering, a M.Sc. in physics, and a Ph.D. in computer science.

Contents: